Logical Agent & Propositional Logic

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References:

1. S. Russell and P. Norvig. *Artificial Intelligence: A Modern Approach*. Chapter 7
2. S. Russell's teaching materials

Introduction

- The representation of *knowledge* and the processes of *reasoning* will be discussed
  - Important for the design of artificial agents
    - Reflex agents
      - Rule-based, table-lookup
    - Problem-solving agents
      - Problem-specific and inflexible
    - Knowledge-based agents
      - Flexible
      - Combine knowledge with current percepts to infer hidden aspects of the current state prior to selecting actions
  - Logic is the primary vehicle for knowledge representation
  - Reasoning copes with different infinite variety of problem states using a finite store of knowledge

Introduction (cont.)

- Example: Natural Language Understanding

John saw the diamond through the window and coveted it

John threw the brick through the window and broke it

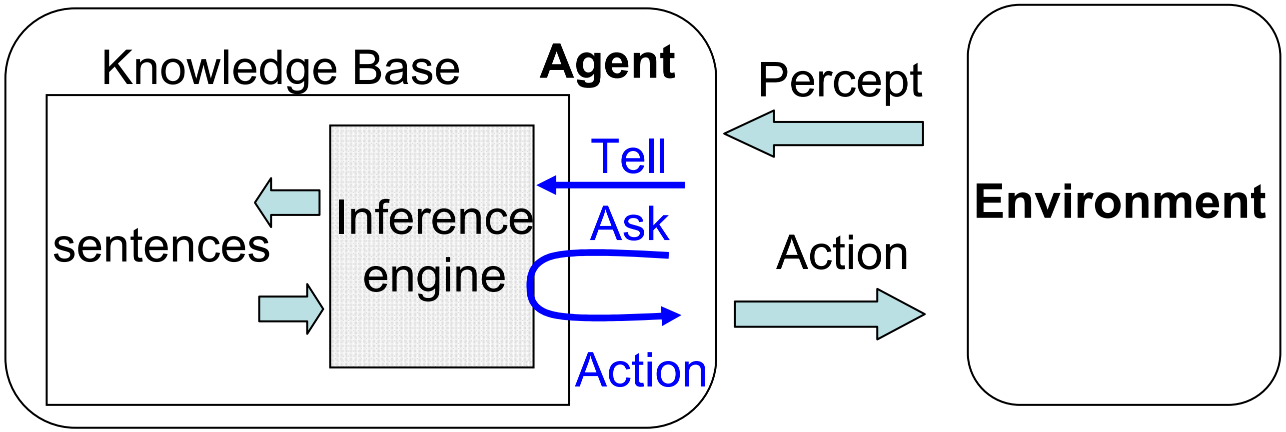
  - Understanding natural language requires inferring the intention of the speaker

Knowledge-Based Agents

- Knowledge base (background knowledge)
  - A set of sentences of formal (or knowledge representation) language
    - Represent facts (assertions) about the world

is a declarative approach

  - Sentences have their syntax and semantics
- Declarative approach to building an agent
  - Tell: tell it what it needs to know (add new sentences to KB)
  - Ask: ask itself what to do (query what is known)



- Inference
  - Derive new sentences from old ones

Knowledge-Based Agents (cont.)

```
function KB-AGENT(percept) returns an action
   static: KB, a knowledge base
           t, a counter, initially 0, indicating time

   TELL(KB, MAKE-PERCEPT-SENTENCE(percept, t))
   action ← ASK(KB, MAKE-ACTION-QUERY(t))
   TELL(KB, MAKE-ACTION-SENTENCE(action, t))
   t ← t + 1
   return action
```

extensive reasoning may be taken here

- KB initially contains some background knowledge
- Each time the agent function is called
  - It Tells KB whit it perceives
  - It Asks KB what action it should perform
- Once the action is chosen
  - The agent records its choice with Tell and executes the action

the internal state

Knowledge-Based Agents (cont.)

- Agents can be viewed at knowledge level
  - What they know, what the goals are, …
- Or agents can be viewed at the implementation level
  - The data structures in KB and algorithms that manipulate them

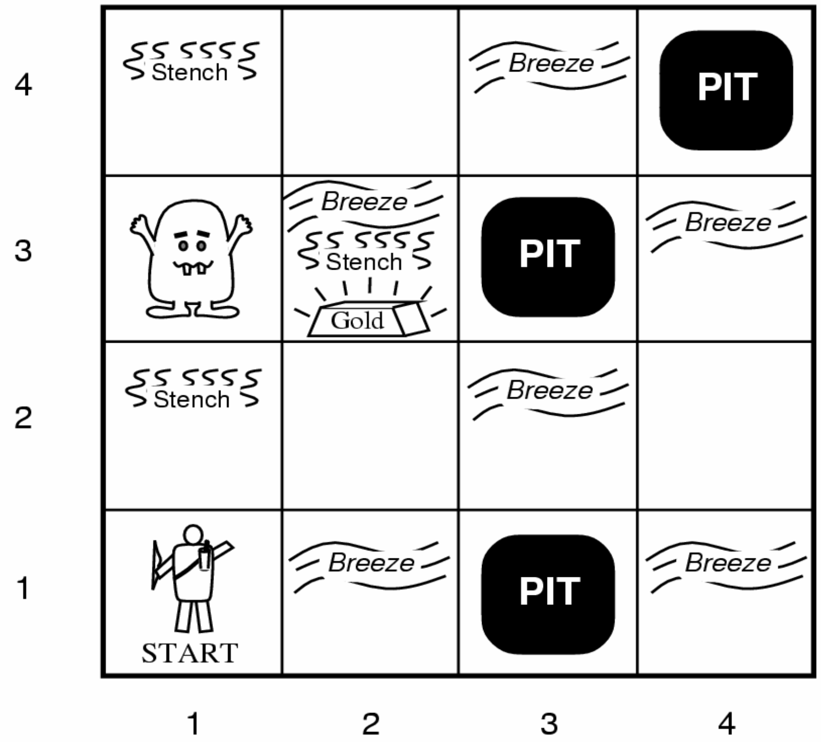
- In summary, the agents must be able to
  - Represent states, actions, etc.
  - Incorporate new percepts
  - Update internal representations of the world
  - Deduce hidden properties of the world
  - Deduce appropriate actions

Wumpus World

- Wumpus world was an early computer game, based on an agent who explores a cave consisting of rooms connected by passageways
- Lurking somewhere in the cave is the wumpus, a beast that eats anyone who enters a room
- Some rooms contain bottomless pits that will trap anyone who wanders into these rooms (except the wumpus, who is too big to fall in)
- The only mitigating features of living in the environment is the probability of finding a heap of gold

Wumpus World PEAS Description

- Performance measure
  - gold +1000, death -1000,
    -1 per step, -10 for using the arrow
- Environment
  - Squares adjacent to wumpus are smelly
  - Squares adjacent to pits are breezy
  - Glitter if gold is in the same square
  - Shooting kills wumpus if you are facing it
  - Shooting uses up the only one arrow
  - Grabbing picks up gold if in same square
  - Releasing drops the gold in same square

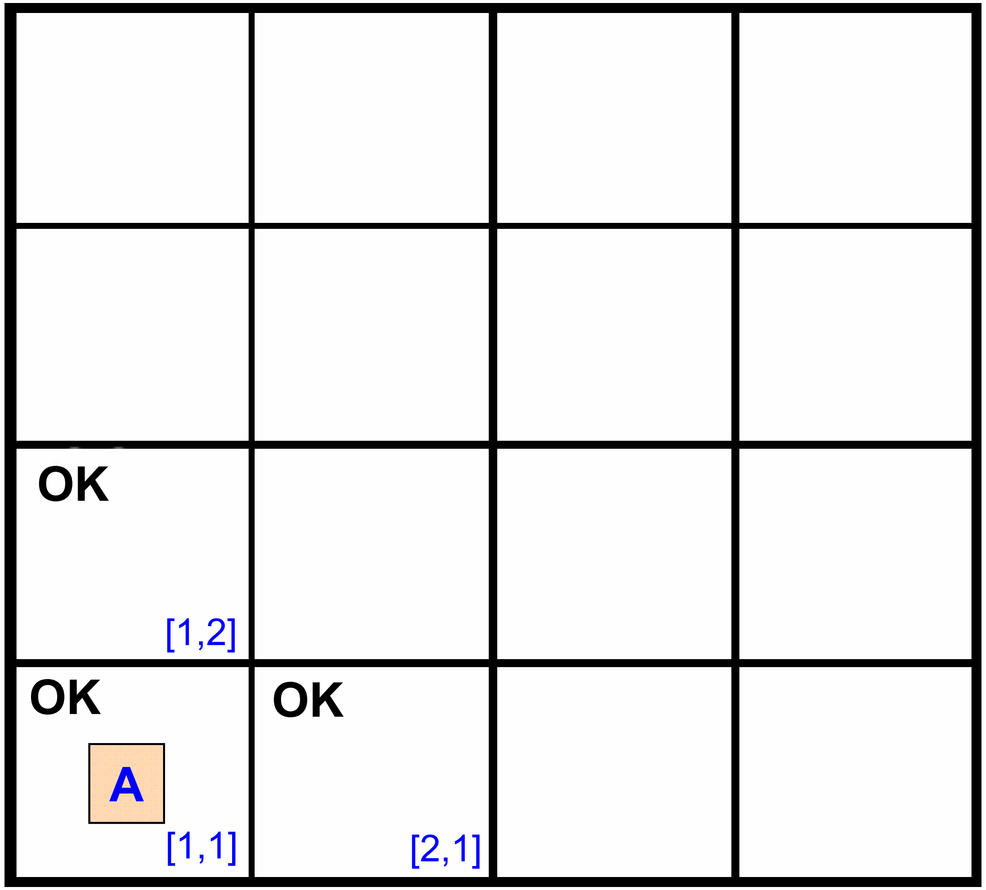


- Actuators
  - Forward, Turn Right, Turn Left, Grab, Release, Shoot
- Sensors
  - Breeze, Glitter, Smell, …

Wumpus World Characterization

- Observable?? No --- only local perception
- Deterministic?? Yes --- outcomes exactly specified
- Episodic?? No --- sequential at the level of actions
- Static?? Yes --- Wumpus and pits can not move
- Discrete?? Yes
- Single-agent?? Yes --- Wumpus is essentially a nature feature

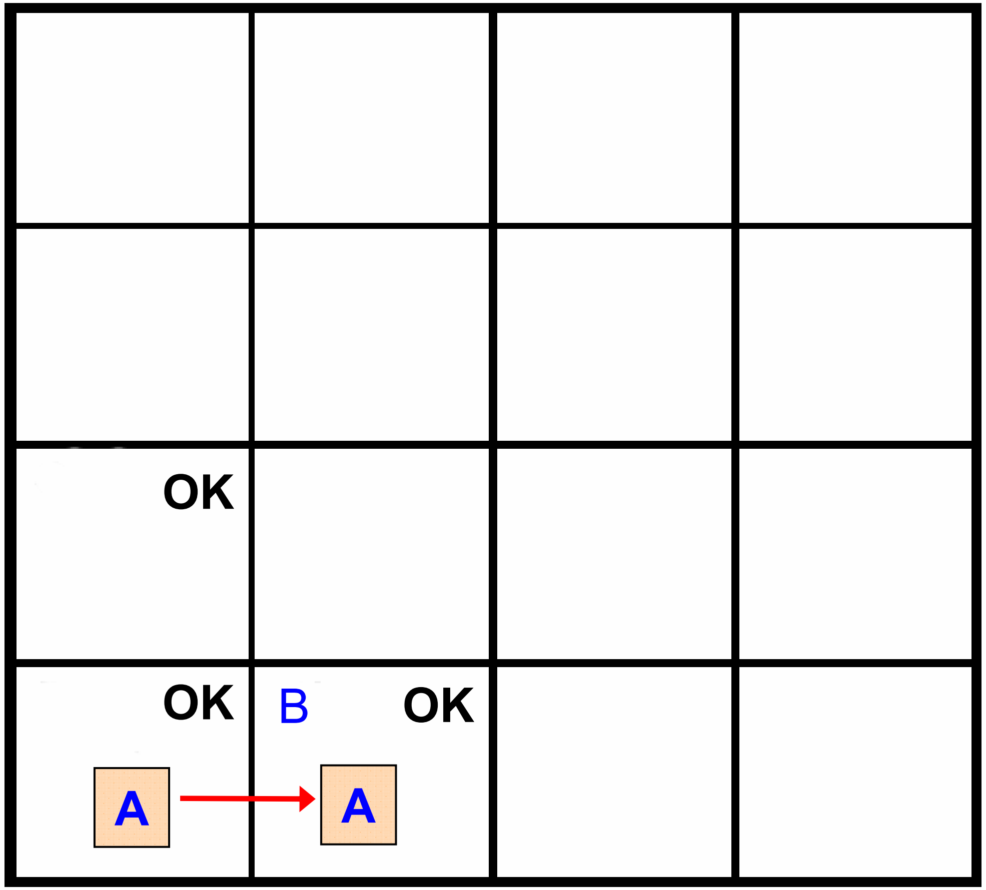
Exploring a Wumpus World



- Initial percept [*None*, *None*, *None*, *None*, *None*]

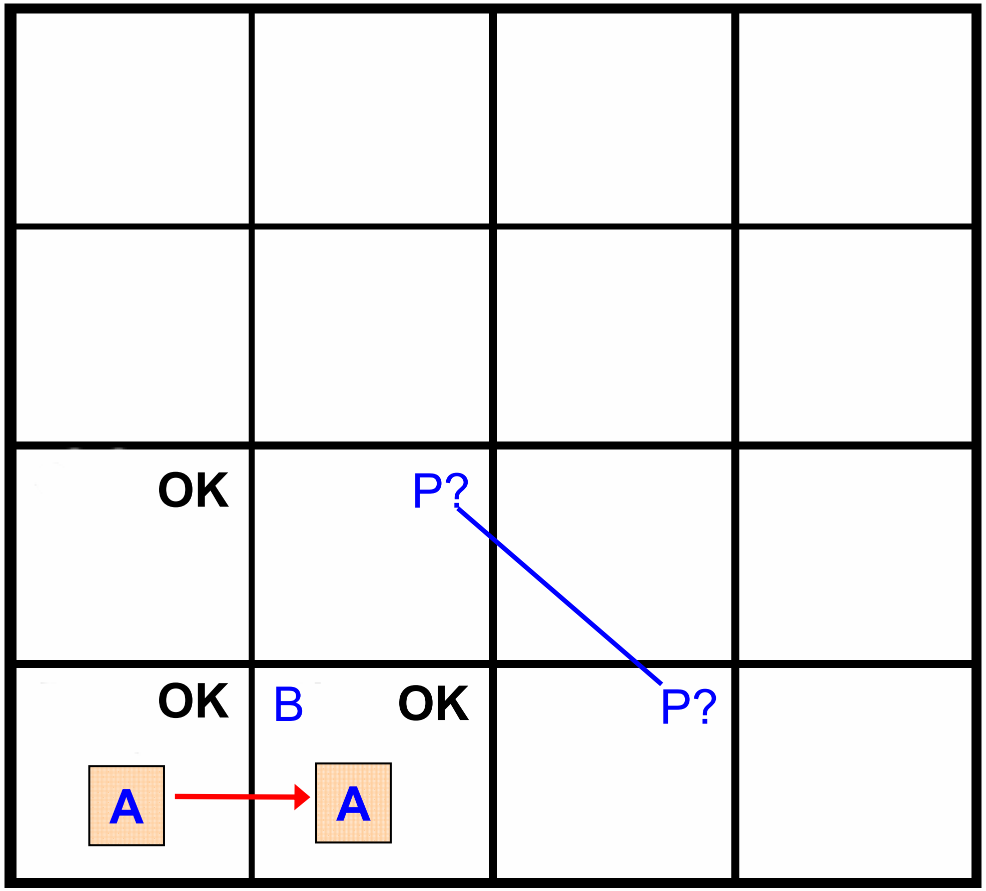
stench breeze glitter bump scream

Exploring a Wumpus World (cont.)

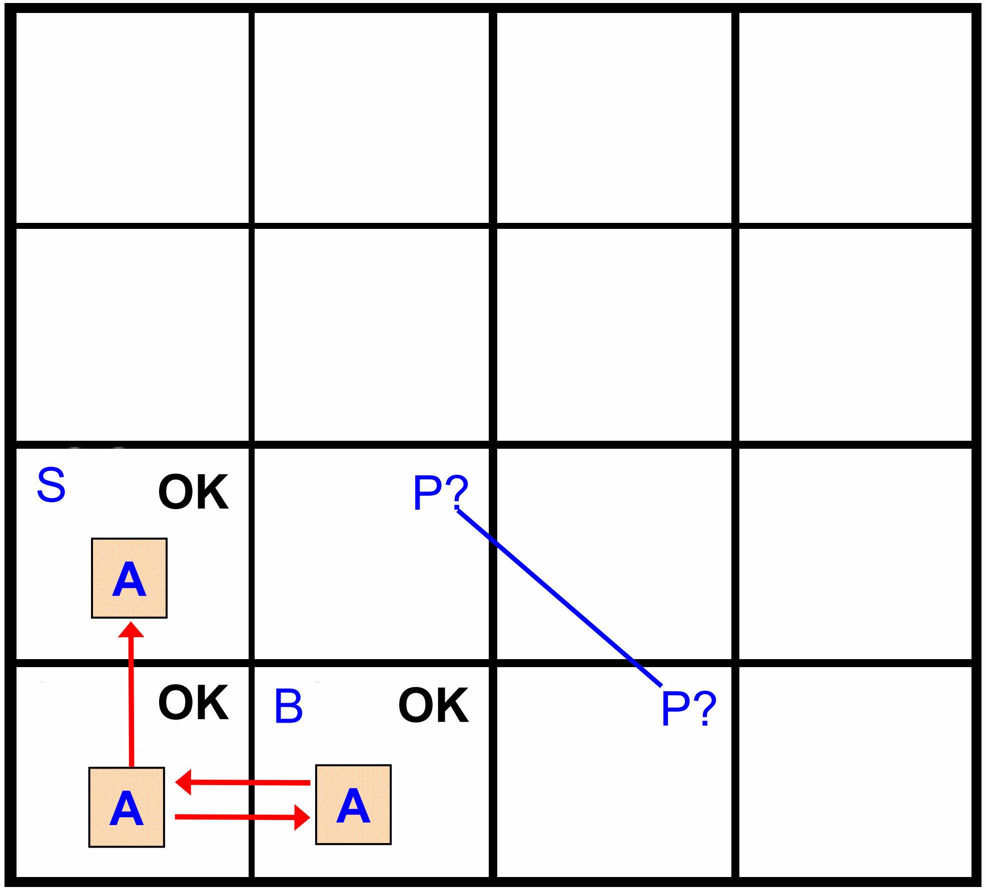


- After the first move, with percept
  [*None, Breeze, None, None, None*]

Exploring a Wumpus World (cont.)

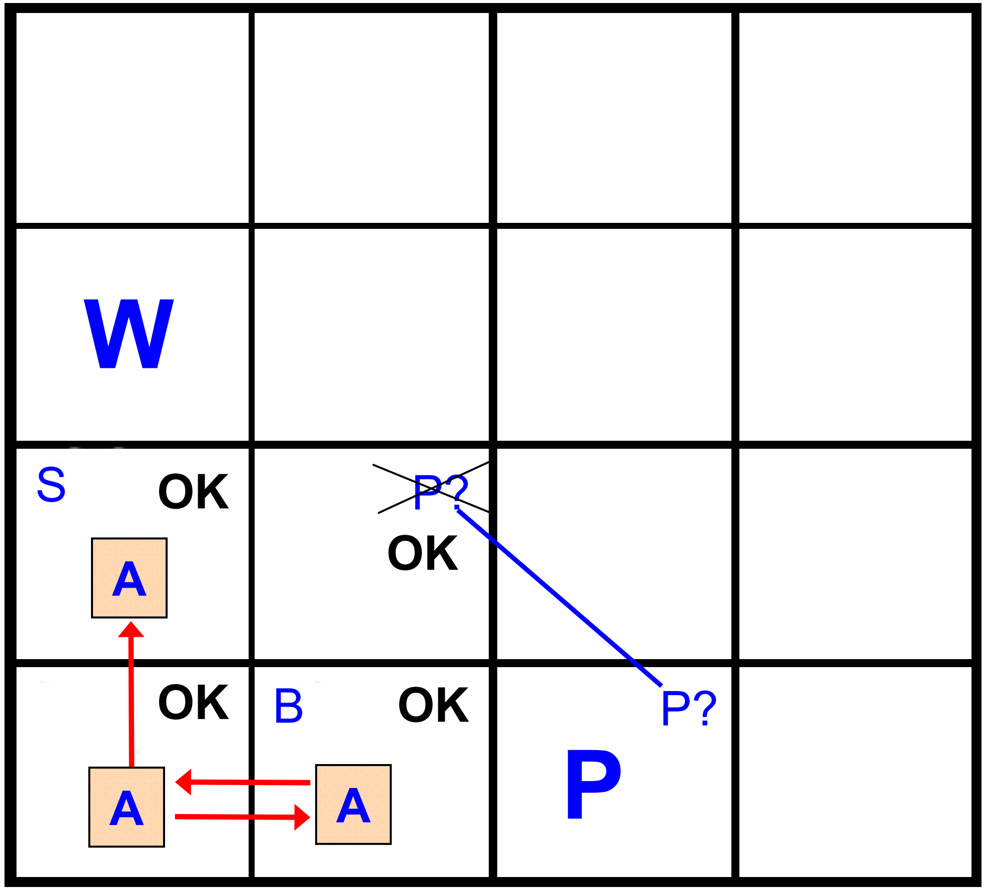


Exploring a Wumpus World (cont.)

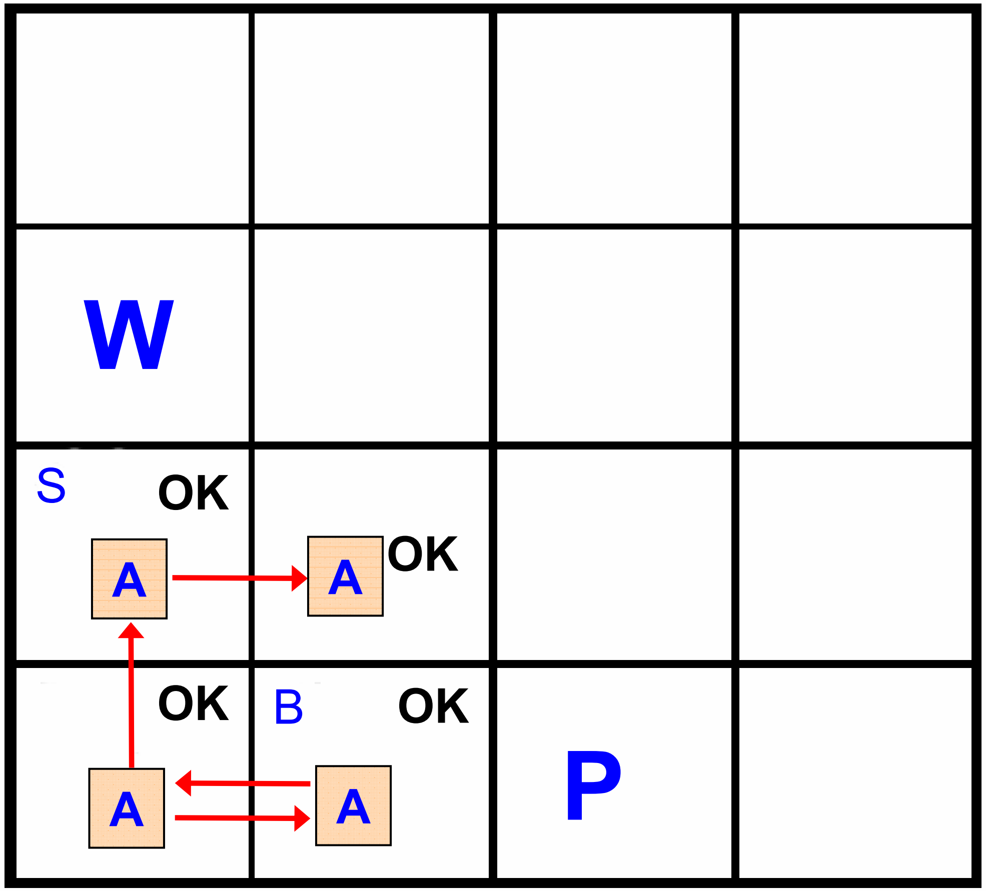


- After the third move, with percept
  [*Stench, None, None, None, None*]

Exploring a Wumpus World (cont.)

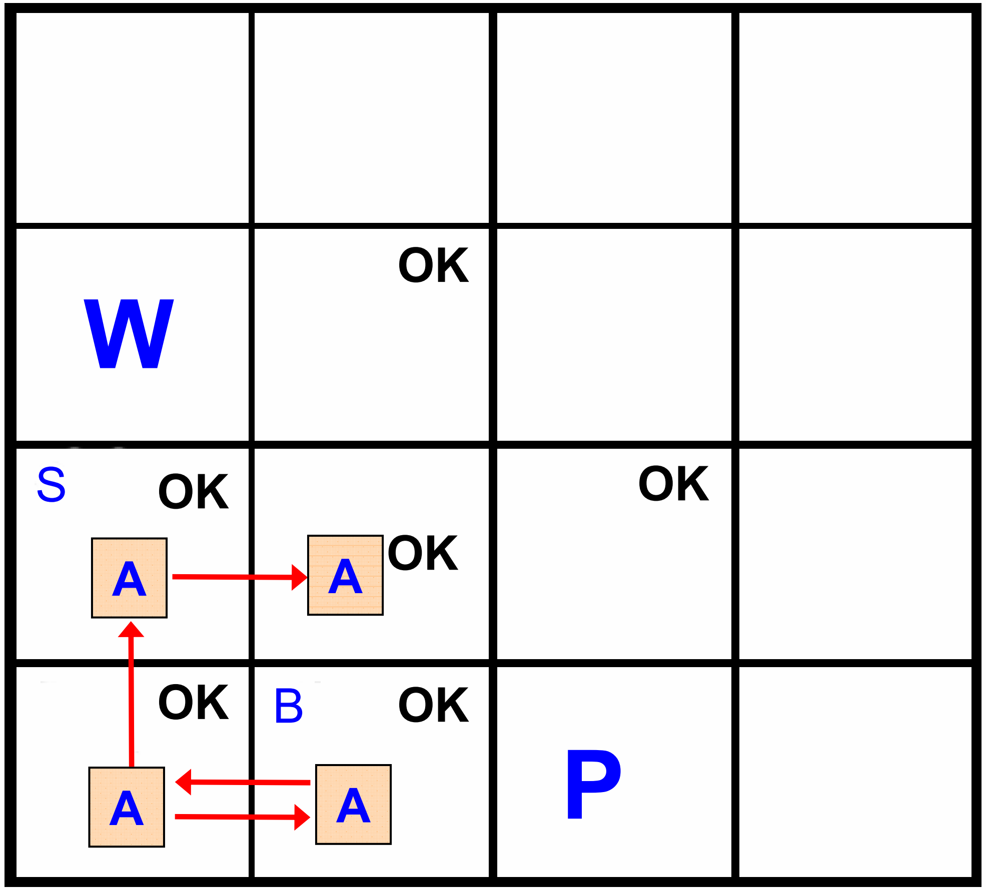


Exploring a Wumpus World (cont.)

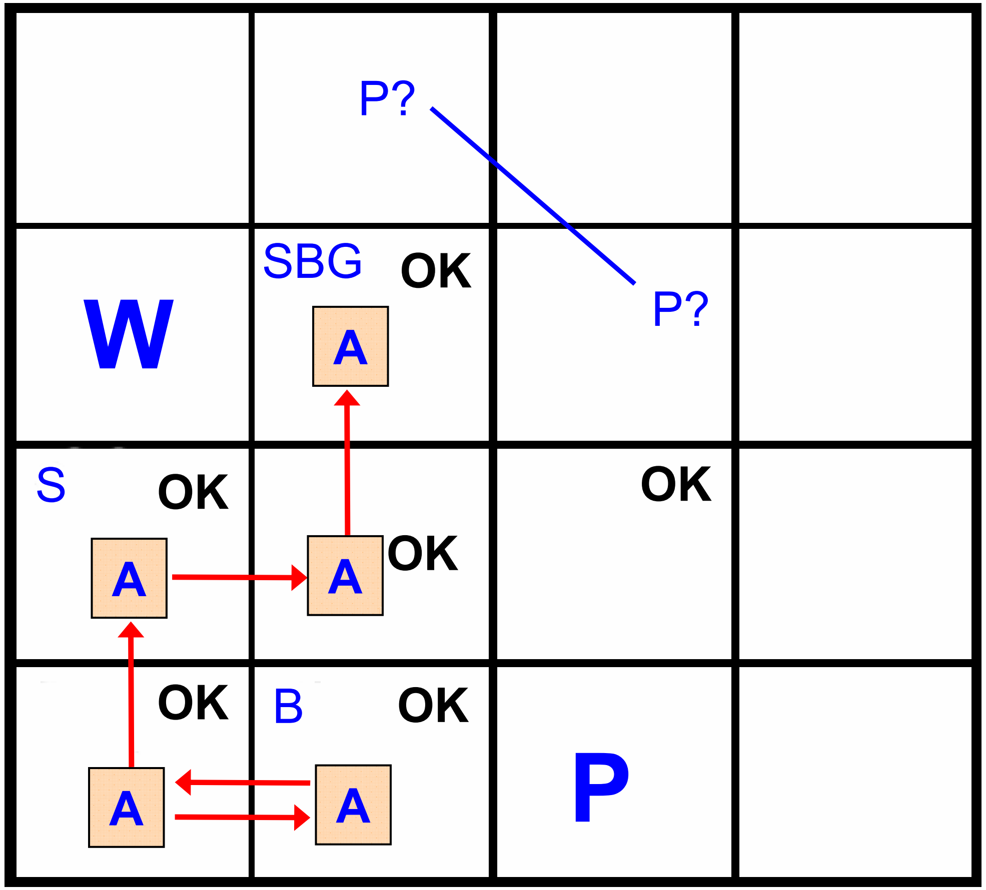


- After the fourth move, with percept
  [*None*, *None*, *None*, *None*, *None*]

Exploring a Wumpus World (cont.)

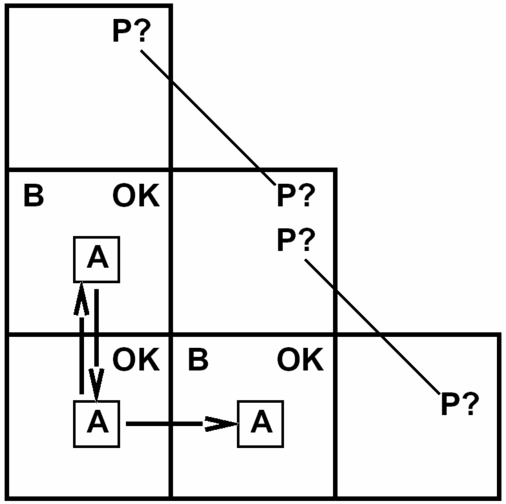


Exploring a Wumpus World (cont.)

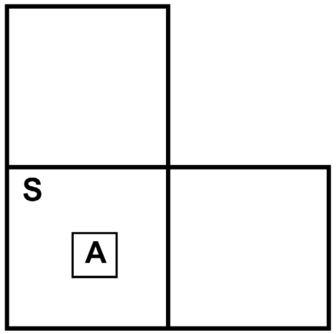


- After the fifth move, with percept
  [*Stench, Breeze, Glitter, None, None*]

Other Tight Spots



Breeze in (1,2) and (2,1)
⇒ No safe actions



Smell in (1,1)
⇒ Cannot move
Can use a strategy of coercion
  shot straight ahead
  wumpus there →dead →safe
  wumpus wasn't there → safe

Logic in General

- Logics are formal languages for representing information such that conclusions can be drawn
- Syntax defines the sentences in the language
- Semantics define the "meaning" of sentences; i.e., define truth or falsehood of a sentence in a world
- E.g., the language of arithmetic

  $x+2 \geq y$ is a sentence; $x2+y>$ is not a sentence

  $x+2 \geq y$ is true iff the number $x+2$ is no less than the number y

  $x+2 \geq y$ is true in a world where $x=7$, $y=1$

  $x+2 \geq y$ is false in a world where $x=0$, $y=6$

  The term "model" will be used to replace the term "world"

- Sentences in an agent's KB are real physical configurations of it

Entailment

- Entailment means that one thing follows from another:

  $KB \models \alpha$

  – Knowledge base *KB* entails sentence α if α is true in all worlds where *KB* is true
    - E.g., the KB containing “the Giants won” and “the Reds won” entails “either the Giants or the Reds won”
    - E.g., $x+y=4$ entails $4=x+y$

  – The knowledge base can be considered as a statement

- Entailment is a relationship between sentences (i.e., syntax) that is based on semantics
  – E.g., $\alpha \models \beta$
    - α entails β
    - $\alpha \models \beta$ iff in every model in which α is true, β is also true
    - Or, if α is true, β must be true

Models

- Logicians typically think in terms of models, which are formally structured worlds with respect to which truth can be evaluated

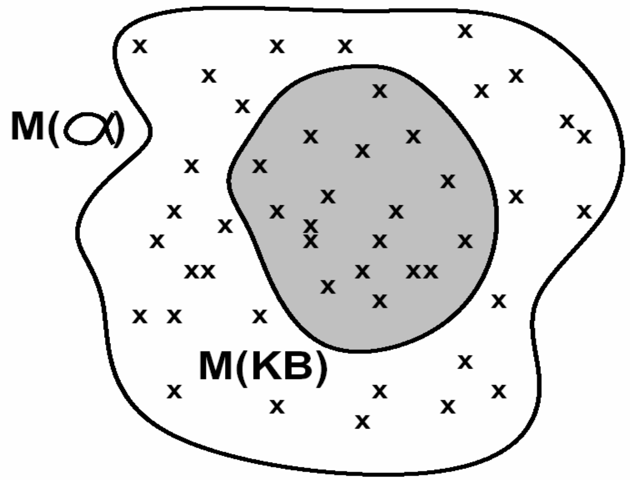
  m is a model of a sentence α iff α is true in *m*

- IF $M(\alpha)$ is the set of all models of α

  Then $KB \models \alpha$ if and only if $M(KB) \subseteq M(\alpha)$

  – I.e., every model in which *KB* is true, α is also true

    • On the other hand, not every model in which α is true, *KB* is also true

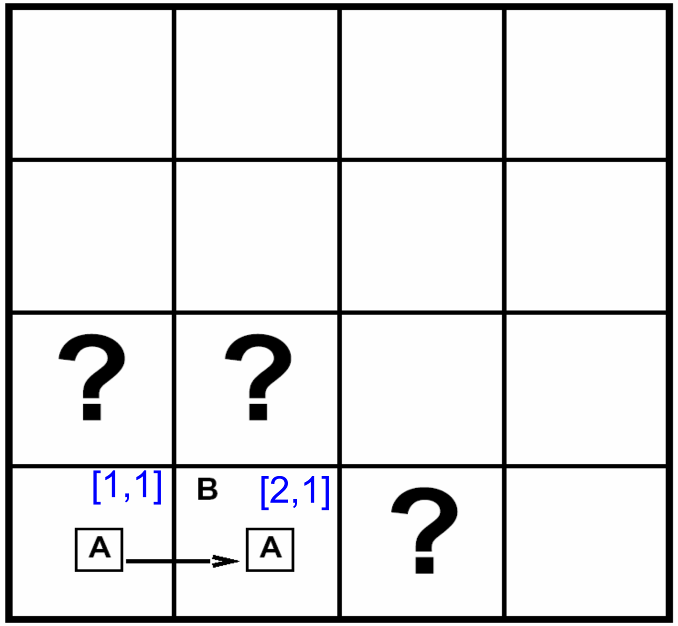


Entailment in the Wumpus World

- Situation after detecting nothing in [1,1], moving right, breeze in [2,1]

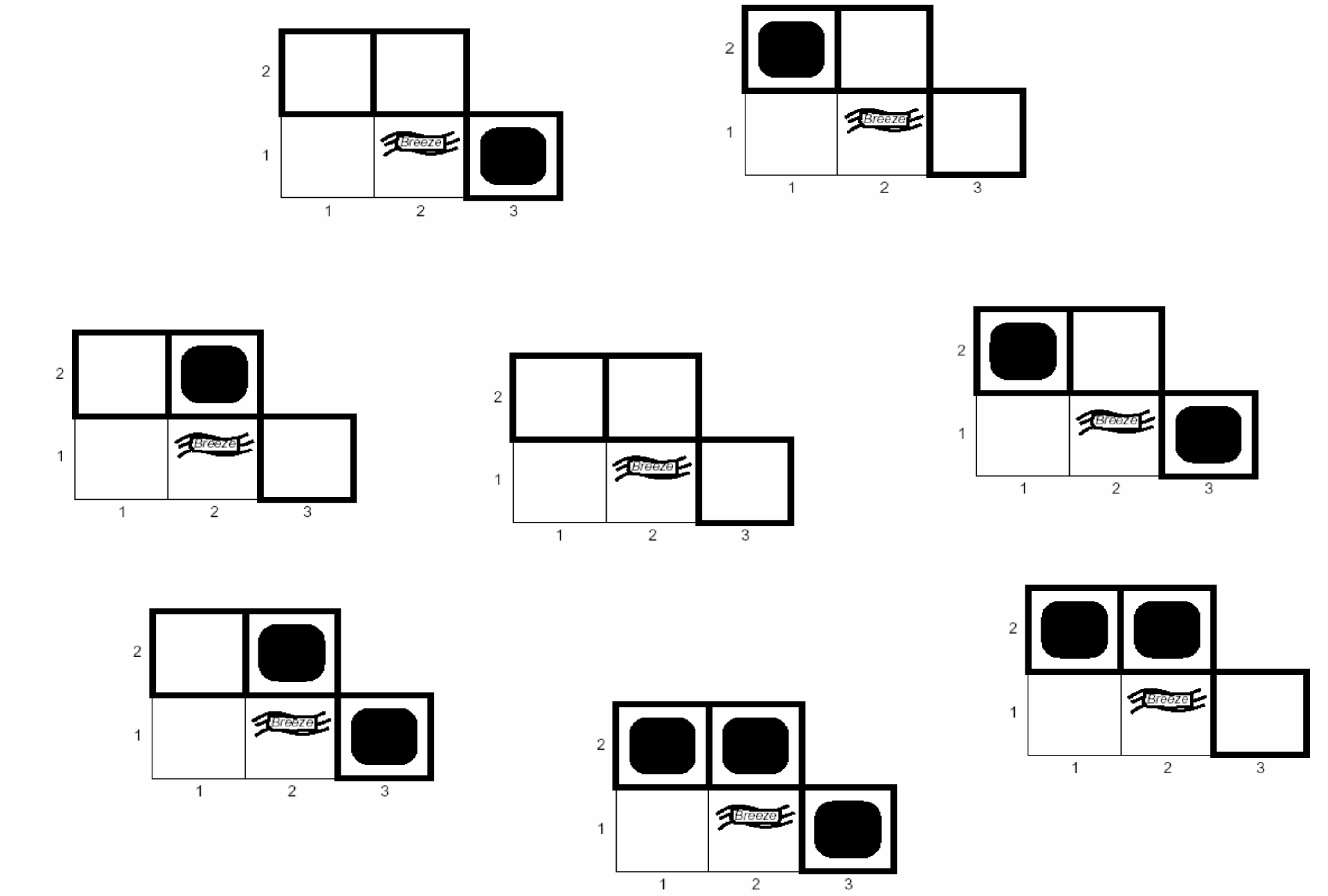
- Consider possible models for **?**s assuming only pits

- 3 Boolean choices $\Rightarrow$ 8 possible models



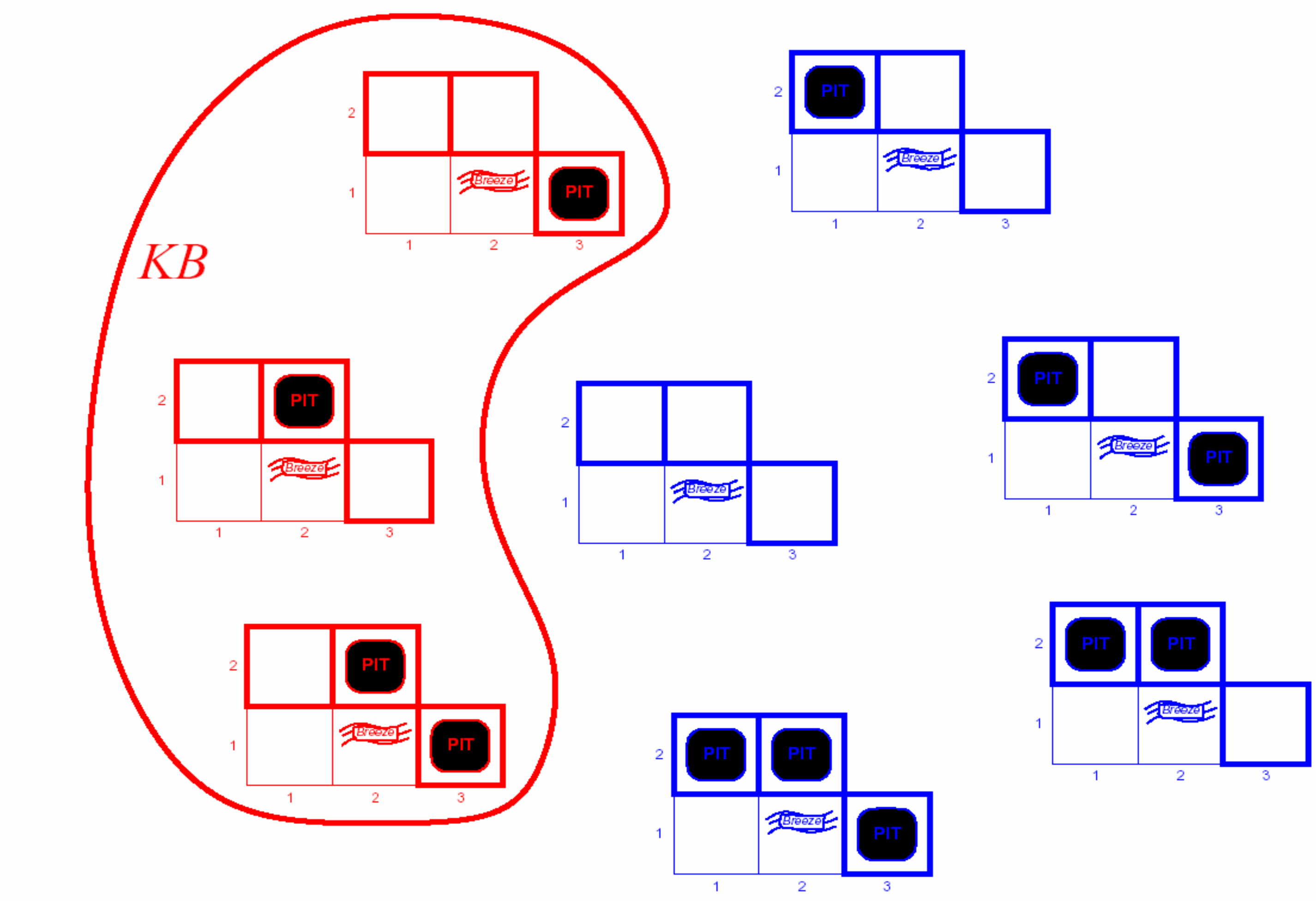
Wumpus Models

- 8 possible models



Wumpus Models (cont.)

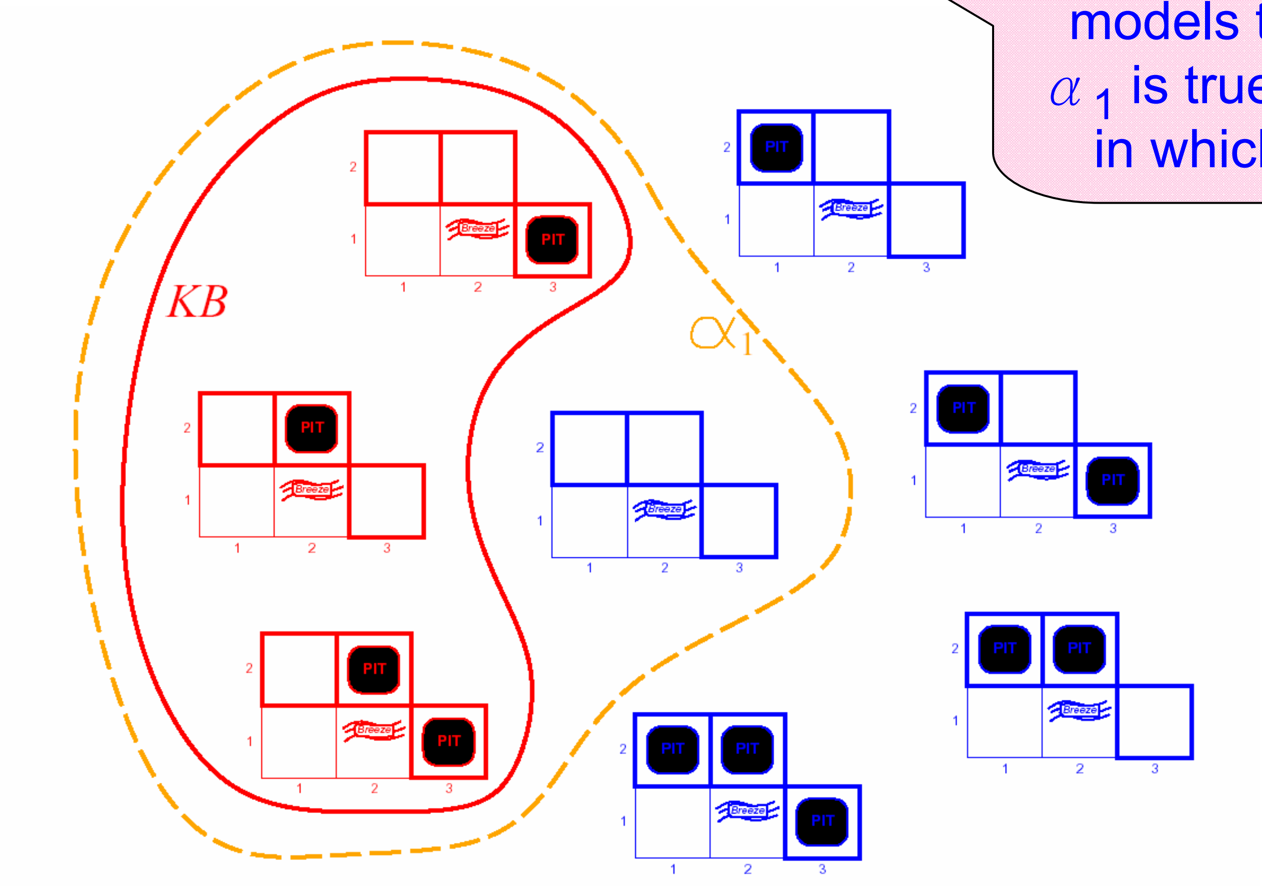
- *KB* = wumpus world-rules + observations



Wumpus Models (cont.)

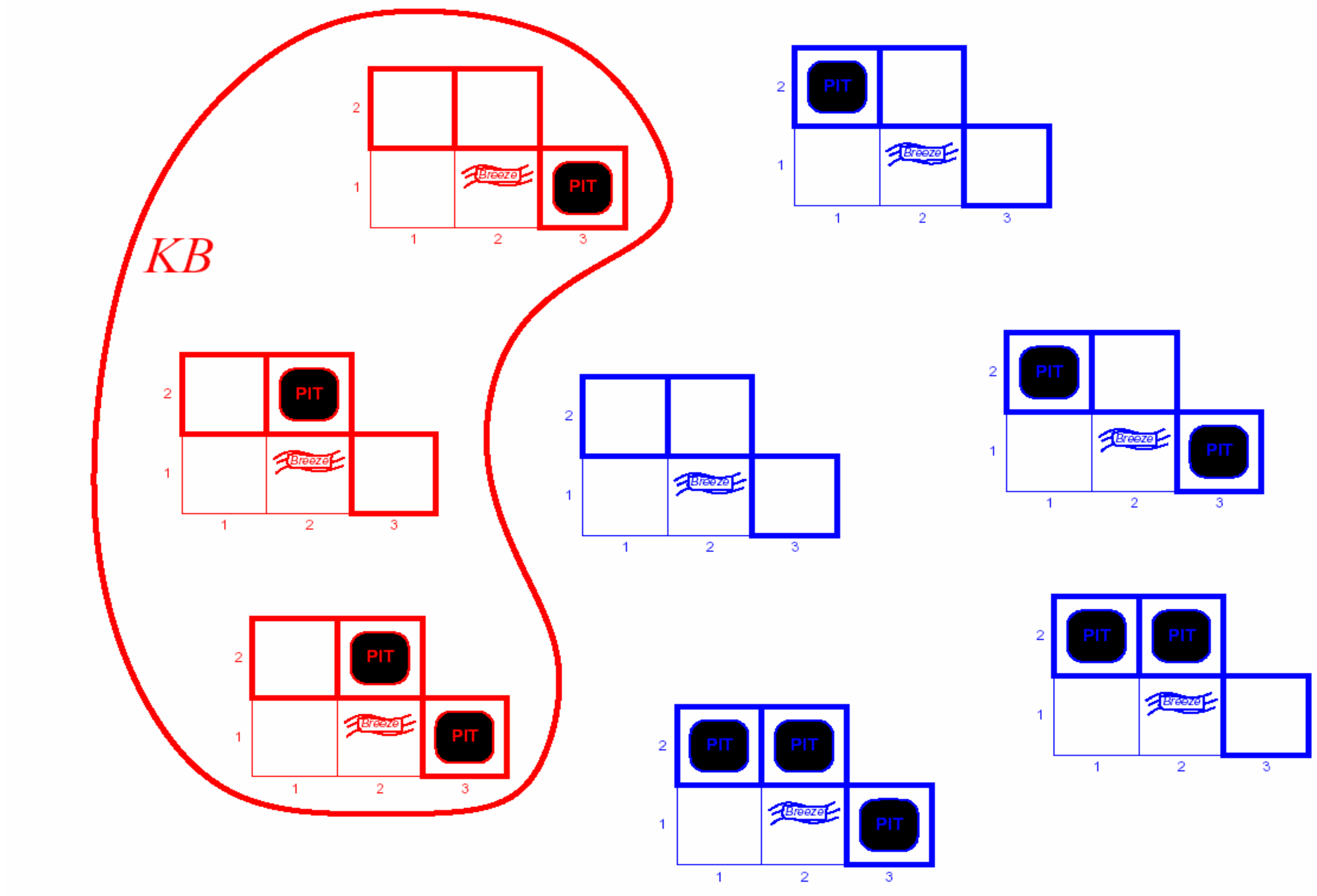
- *KB* = wumpus world-rules + observations
  - α_1= “[1,2] is safe” (no pit in [1, 2])
  - *KB* |= α_1, proved by model checking

enumerate all possible models to check that α_1 is true in all models in which *KB* is true



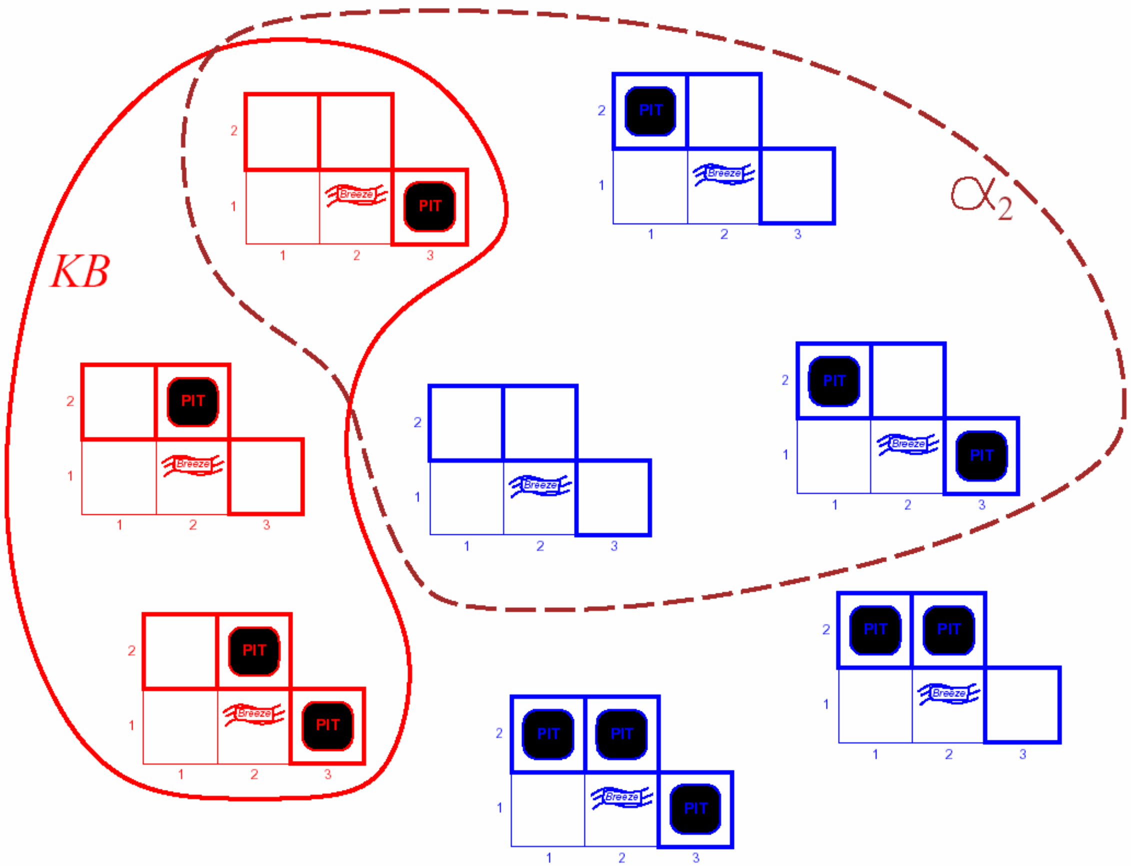
Wumpus Models (cont.)

- *KB* = wumpus world-rules + observations



Wumpus Models (cont.)

- *KB* = wumpus world-rules + observations
  - α_2= "[2,2] is safe" (no pit in [2, 2])
  - *KB* $|\neq \alpha_2$, proved by model checking



Inference

- $KB \vdash_i \alpha$
  - Sentence α can be derived from *KB* by inference algorithm *i*
  - Think of

    the set of all consequences of KB as a haystack

    α as a needle

    entailment like the needle in the haystack

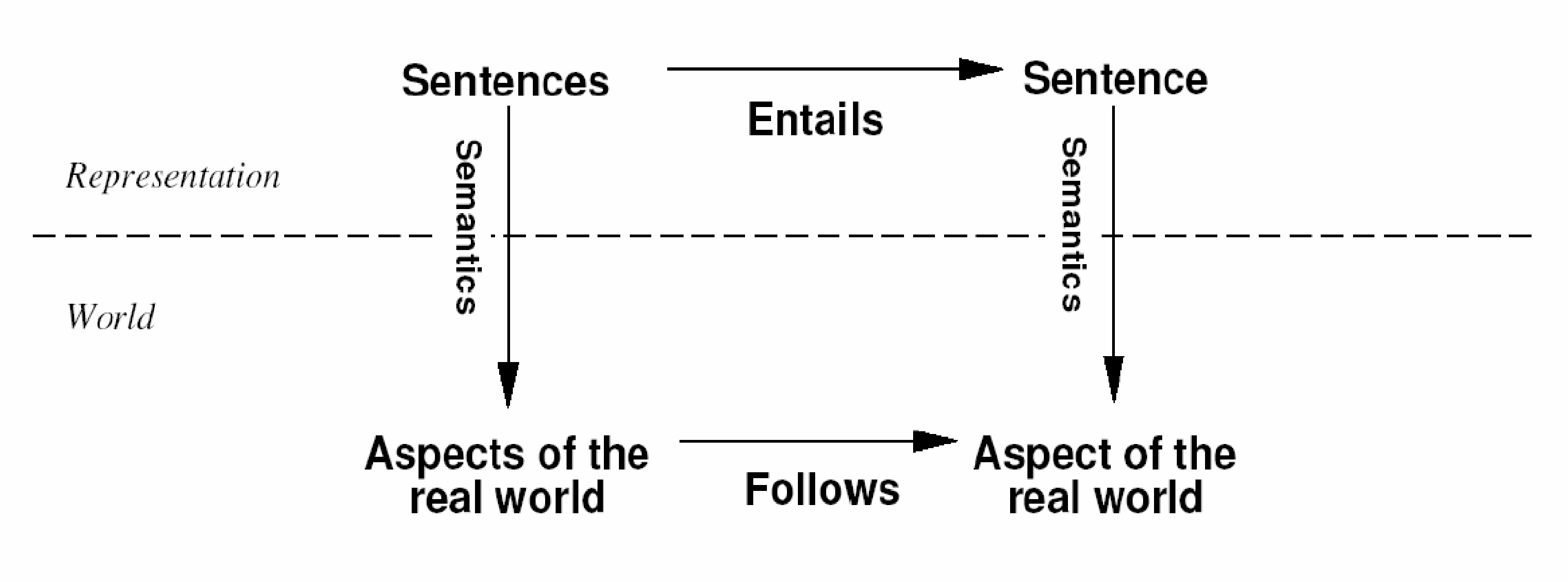
    inference like finding it

- Soundness or truth-preserving inference
  - An algorithm *i* is sound if whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$
  - That is the algorithm derives only entailed sentences
  - The algorithm won't announce " the discovery of nonexistent needles"

Inference (cont.)

- Completeness
  - An algorithm *i* is complete if whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$
  - A sentence α will be generated by an inference algorithm *i* if it is entailed by the *KB*
  - Or says, the algorithm will answer any question whose answer follows from what is known by the *KB*

Inference (cont.)



– Sentences are physical configurations of the agent, and reasoning is a process of constructing new physical configurations from old ones

– Logical reasoning should ensure that the new configurations represent aspects of the world that actually follow from the aspects that the old configurations represent

Propositional Logic: Syntax

- Propositional logic is the simplest logic that illustrates basic ideas

- Syntax: defines the allowable sentences
  - Atomic sentences consist of a single propositional symbols
  - Propositional symbols: e.g., *P, Q* and *R*
    - Each stands for a proposition (fact) that can be either true or false
  - Complex sentences are constructed from simpler one using logic connectives

    $\wedge$ (and) conjunction

    $\vee$ (or) disjunction

    $\Rightarrow$ (implies) implication

    $\Leftrightarrow$ (equivalent) equivalence, or biconditional

    $\neg$ (not) negation

Propositional Logic: Syntax (cont.)

- **BNF** (Backus-Naur Form) grammar for propositional logic

$$
\begin{aligned}
\mathit{Sentence} &\rightarrow \mathit{Atomic\ Sentence} \mid \mathit{Complex\ Sentence} \\
\mathit{Atomic\ Sentence} &\rightarrow \text{True} \mid \text{False} \mid \mathit{Symbol} \\
\mathit{Symbol} &\rightarrow P \mid Q \mid R \ldots \\
\mathit{Complex\ Sentence} &\rightarrow \neg\, \mathit{Sentence} \\
&\mid (\mathit{Sentence} \wedge \mathit{Sentence}) \\
&\mid (\mathit{Sentence} \vee \mathit{Sentence}) \\
&\mid (\mathit{Sentence} \Rightarrow \mathit{Sentence}) \\
&\mid (\mathit{Sentence} \Leftrightarrow \mathit{Sentence})
\end{aligned}
$$

- Order of precedence: (from highest to lowest)

  $\neg$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$

  – E.g., $\neg \mathrm{P} \vee Q \wedge R \Rightarrow S$ means $((\neg \mathrm{P}) \vee (Q \wedge R)) \Rightarrow S$

    $A \Rightarrow B \Rightarrow C$ is not allowed !

Propositional Logic: Semantics

- Define the rules for determining the truth of a sentence with respect to a particular model
  - Each model fixes the truth value (true or false) for every propositional symbol
  - E.g., $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
    - 3 symbols, 8 possible models, can be enumerated automatically
    - A possible model m_1 $\{P_{1,2}= \textit{false}, P_{2,2}=\textit{false}, P_{3,1}= \textit{true}\}$
  - Simple recursive process evaluates an arbitrary sentence, e.g.,

$$\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1}) = \textit{true} \wedge (\textit{false} \vee \textit{true}) = \textit{true} \wedge \textit{true} = \textit{true}$$

Models for PL are just sets of truth values for the propositional symbols

Truth Tables for Connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

$\neg P$	is true iff	P	is false			
$P \wedge Q$	is true iff	P	is true	and	Q	is true
$P \vee Q$	is true iff	P	is true	or	Q	is true
$P \Rightarrow Q$	is false iff	P	is true	and	Q	is false
$P \Leftrightarrow Q$	is true iff	$P \Rightarrow Q$	is true	and	$Q \Rightarrow P$	is true

$$P \Rightarrow Q$$

Premise, Body — Conclusion, Head

More about Implication

- For an implication: $P \Rightarrow Q$
  - Which doesn't need any relation of causation or relevance between P and Q
    - " 5 is odd implies Tokyo is the capital of Japan" is true

- We can think of "$P \Rightarrow Q$" as saying
  - If P is true, then I am claiming that Q is true. Otherwise I am making no claim

Knowledge Base

- Knowledge base, consisting of a set of sentences, can be considered as a single sentence
  - A conjunction of these sentences
  - Knowledge base asserts that all the individual sentences are true

Wumpus World Sentences

- Let $P_{i,j}$ be true if there is a pit in [*i, j*]
- Let $B_{i,j}$ be true if there is a breeze in [*i, j*]
- A square is breezy *if only if* there is an adjacent pit

R_1: $\neg P_{1,1}$ — no pit in [1,1]

R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$ — pits cause breezes in adjacent squares

R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

R_4: $\neg B_{1,1}$ — no breeze in [1,1]

R_5: $B_{2,1}$ — breeze in [2,1]

- Note: there are 7 proposition symbols involved
  - $B_{1,1}, B_{2,1}, P_{1,1}, P_{1,2}, P_{2,1}, P_{2,2}, P_{3,1}$
  - There are 2^7=128 models !
    - While only three of them satisfy the above 5 descriptions/sentences

Truth Tables for Inference

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	KB	α_1
false	*false*	*false*	*false*	*false*	*false*	*false*	*false*	*true*
false	*false*	*false*	*false*	*false*	*false*	*true*	*false*	*true*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
false	*true*	*false*	*false*	*false*	*false*	*false*	*false*	*true*
false	*true*	*false*	*false*	*false*	*false*	*true*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*false*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*true*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*true*	*false*	*false*	*false*	*true*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
true	*true*	*true*	*true*	*true*	*true*	*true*	*false*	*false*

128 models

$R_1 \wedge R_2 \wedge R_3 \wedge R_4 \wedge R_5$

Conjunction of sentences of KB

$\neg P_{1,2}$

- $P_{2,2}$?

Inference by Enumeration (Model Checking)

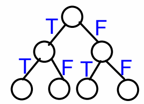
Test if KB is true α is also true

```
function TT-ENTAILS?(KB, α) returns true or false
   inputs: KB, the knowledge base, a sentence in propositional logic
           α, the query, a sentence in propositional logic

   symbols ← a list of the proposition symbols in KB and α
   return TT-CHECK-ALL(KB, α, symbols, [ ])
--------------------------------------------------------------------
function TT-CHECK-ALL(KB, α, symbols, model) returns true or false
   if EMPTY?(symbols) then
      if PL-TRUE?(KB, model) then return PL-TRUE?(α, model)
      else return true
   else do
      P ← FIRST(symbols); rest ← REST(symbols)
      return TT-CHECK-ALL(KB, α, rest, EXTEND(P, true, model)) and
             TT-CHECK-ALL(KB, α, rest, EXTEND(P, false, model))
```

Implement the definition of entailment

(if not a model for KB→don't care)

Return a new partial model in which *P* has the value *true*



- A recursive depth-first enumeration of all models (assignments to variables)
  - Sound and complete
  - Time complexity: $O(2^n)$ exponential in the size of the input
  - Space complexity: $O(n)$

Logical Equivalences

- Two sentences are logically equivalent iff true in same set of models

entailment

$$\alpha \equiv \beta \quad \text{if and only if} \quad \alpha \models \beta \text{ and } \beta \models \alpha$$

$$M(\alpha) \subseteq M(\beta) \text{ and}$$
$$M(\beta) \subseteq M(\alpha)$$

$$\therefore M(\beta) = M(\alpha)$$

Logical Equivalences (cont.)

$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$	commutativity of $\wedge$
$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$	commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma))$	associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma))$	associativity of $\vee$
$\neg(\neg\alpha) \equiv \alpha$	double-negation elimination
$(\alpha \Rightarrow \beta) \equiv (\neg\beta \Rightarrow \neg\alpha)$	contraposition
$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \vee \beta)$	implication elimination
$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$	biconditional elimination
$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$	De Morgan
$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$	De Morgan
$(\alpha \wedge (\beta \vee \gamma)) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	distributivity of $\vee$ over $\wedge$

Validity and Satisfiability

- A sentence is valid (or tautological) if it is true in all models

  $True,\ A \lor \neg A,\ A \Rightarrow A,\ (A \land (A \Rightarrow B)) \Rightarrow B$

- Validity is connected to inference via Deduction Theorem:

  $KB \models \alpha$ if only if $(KB \Rightarrow \alpha)$ is valid

- A sentence is satisfiable if it is true in some model

  $A,\ B \land \neg C$

- A sentence is unsatifiable if it is true in no models

  $A \land \neg A$

- Satisfiablity is connected to inference via refutation (or proof by contradiction)

  $KB \models \alpha$ if only if $(KB \land \neg \alpha)$ is unsatifiable

Determination of satisfiability of sentences in PL is NP-complete

Patterns of Inference: Inference Rules

- Applied to derive chains of conclusions that lead to the desired goal
- Modus Ponens (Implication Elimination, *if-then* reasoning)

$$\frac{\alpha \Rightarrow \beta, \qquad \alpha}{\beta}$$

- And Elimination

$$\frac{\alpha \wedge \beta}{\alpha}$$

- Biconditional Elimination

$$\frac{\alpha \Leftrightarrow \beta}{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)} \quad \text{and} \quad \frac{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}{\alpha \Leftrightarrow \beta}$$

Patterns of Inference: Inference Rules (cont.)

- Example
  - With the KB as the following, show that $\neg P_{1,2}$

R_1: $\neg P_{1,1}$	no pit in [1,1]
R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$	pits cause breezes in adjacent squares
R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$	
R_4: $\neg B_{1,1}$	no breeze in [1,1]
R_5: $B_{2,1}$	breeze in [2,1]

1. Apply biconditional elimination to R_2

   R_6: $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

2. Apply And-Elimination to R_6

   R_7: $(P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}$

3. Logical equivalence for contrapositives

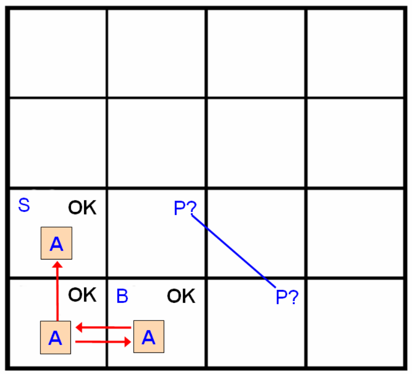
   R_8: $\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1})$

4. Apply Modus Ponens with R_8 and the percept R_4

   R_9: $\neg(P_{1,2} \vee P_{2,1})$

5. Apply De Morgan's rule and give the conclusion

   R_{10}: $\neg P_{1,2} \wedge \neg P_{2,1}$



6. Apply And-Elimination to R_{10}

   R_{11}: $\neg P_{1,2}$

Patterns of Inference: Inference Rules (cont.)

- Unit Resolution

$$\frac{\alpha \vee \beta, \quad \neg \beta}{\alpha}$$

$$\frac{l_1 \vee l_2 \vee \cdots \vee l_k, \quad m}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k}$$

l_i and m are complementary literals

- Resolution

Resolution is used to either confirm or refute a sentence, but it can't be used to enumerate sentences

$$\frac{\alpha \vee \beta, \quad \neg \beta \vee \gamma}{\alpha \vee \gamma}$$

$$\frac{l_1 \vee l_2 \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_n}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k \vee m_1 \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

l_i and m_j are complementary literals

  - E.g., $$\frac{P_{1,1} \vee P_{3,1}, \quad \neg P_{1,1} \vee \neg P_{2,2}}{P_{3,1} \vee \neg P_{2,2}}$$

  - Multiple copies of literals in the resultant clause should be removed (such a process is called factoring)

Patterns of Inference: Inference Rules (cont.)

- Unit Resolution

$$\alpha \vee \beta \text{ is true } \quad (\alpha \vee \beta \equiv \neg \beta \Rightarrow \alpha)$$
$$\text{and } \quad \neg \beta \text{ is true}$$
$$\therefore \alpha \quad \text{is true}$$

- Resolution

$$\alpha \vee \beta \text{ is true } \quad (\alpha \vee \beta \equiv \neg \beta \Rightarrow \alpha)$$
$$\text{and } \quad \neg \beta \vee \gamma \text{ is true } \quad (\neg \beta \vee \gamma \equiv \neg \gamma \Rightarrow \neg \beta)$$
$$\therefore \neg \gamma \Rightarrow \alpha \quad (\neg \gamma \Rightarrow \alpha \equiv \alpha \vee \gamma)$$

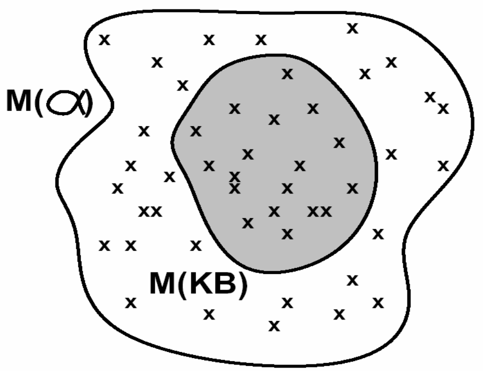
Monotonicity

- The set of entailed sentences can only increase as information is added to the knowledge base

If $KB \models \alpha$ then $KB \wedge \beta \models \alpha$

  - The additional assertion β can't invalidate any conclusion α already inferred
  - E.g., α : there is not pit in [1,2]

    β : there is eight pits in the world



Normal Forms

- Conjunctive Normal Form (*CNF*)
  - A sentence expressed as a conjunction of disjunctions of literals
  - E.g., $(P \vee Q) \wedge (\neg P \vee R) \wedge (\neg S)$

- Also, Disjunction Normal Form (*DNF*)
  - A sentence expressed as a disjunction of conjunctions of literals
  - E.g., $(P \wedge Q) \vee (\neg P \wedge R) \vee (\neg S)$

- An arbitrary propositional sentence can be expressed in *CNF* (or *DNF*)

Normal Forms (cont.)

- Example: convert $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$ into *CNF*

1. Eliminate $\Leftrightarrow$, replace $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$

   $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

2. Eliminate $\Rightarrow$, replace $\alpha \Rightarrow \beta$ with $(\neg \alpha \vee \beta)$

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg (P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

3. Move $\neg$ inwards

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$

4. Apply distributivity law

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

Resolution Algorithm

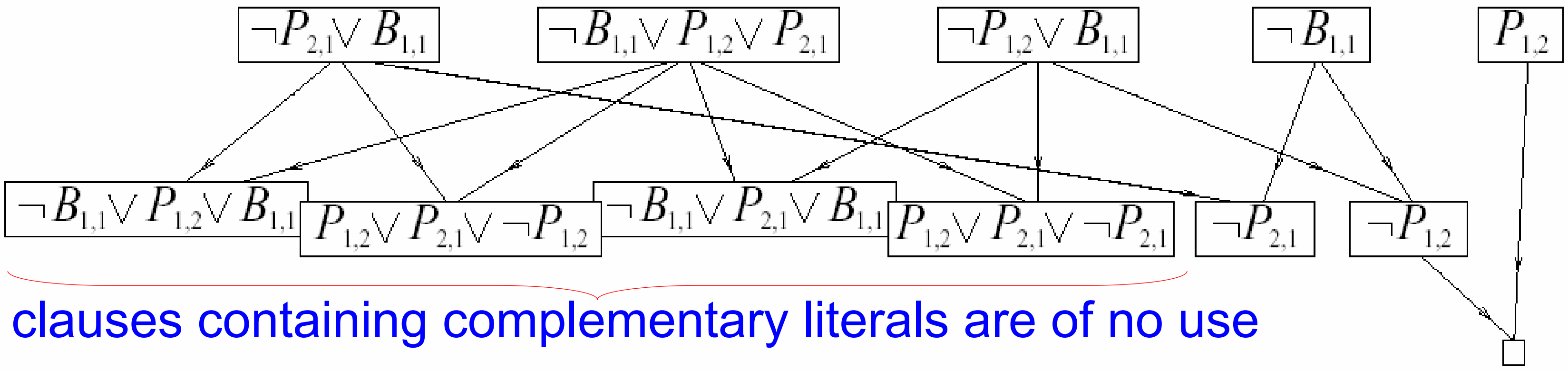
```
function PL-RESOLUTION(KB, α) returns true or false
   inputs: KB, the knowledge base, a sentence in propositional logic
           α, the query, a sentence in propositional logic

   clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
   new ← { }
   loop do
      for each C_i, C_j in clauses do
         resolvents ← PL-RESOLVE(C_i, C_j)
         if resolvents contains the empty clause then return true
         new ← new ∪ resolvents
      if new ⊆ clauses then return false //no new clauses resolved
      clauses ← clauses ∪ new
```

proof by contradiction

- To show that *KB* |= α , we show that *(KB* $\wedge \neg \alpha$) is unsatisfiable
- Each pair that contains complementary literals is resolved to produce new clause until one of the two things happens:
  (1) No new clauses can be added ⇨ *KB* does not entail α
  (2) Empty clause is derived ⇨ *KB* entails α

Resolution Example

$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$ $\alpha = \neg P_{1,2}$



- Empty clause – disjunction of no disjuncts
  - Equivalent to false
  - Represent a contradiction here

$(B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$

$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1}) \wedge \neg B_{1,1}$

We have shown it before !

Horn Clauses

- A Horn clause is a disjunction of literals of which at most one is positive
  - E.g., $\neg P_1 \vee \neg P_2 \vee \ \dots \ \vee \neg P_n \vee Q$
- Every Horn clause can be written as an implication
  - The premise is a conjunction of positive literals
  - The conclusion is a single positive literal
  - E.g., $\neg P_1 \vee \neg P_2 \vee \ \dots \ \vee \neg P_n \vee Q$ can be converted to $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \Rightarrow Q$
- Inference with Horn clauses can be done naturally through the forward chaining and backward chaining, which be will be discussed later on
  - The application of Modus Ponens

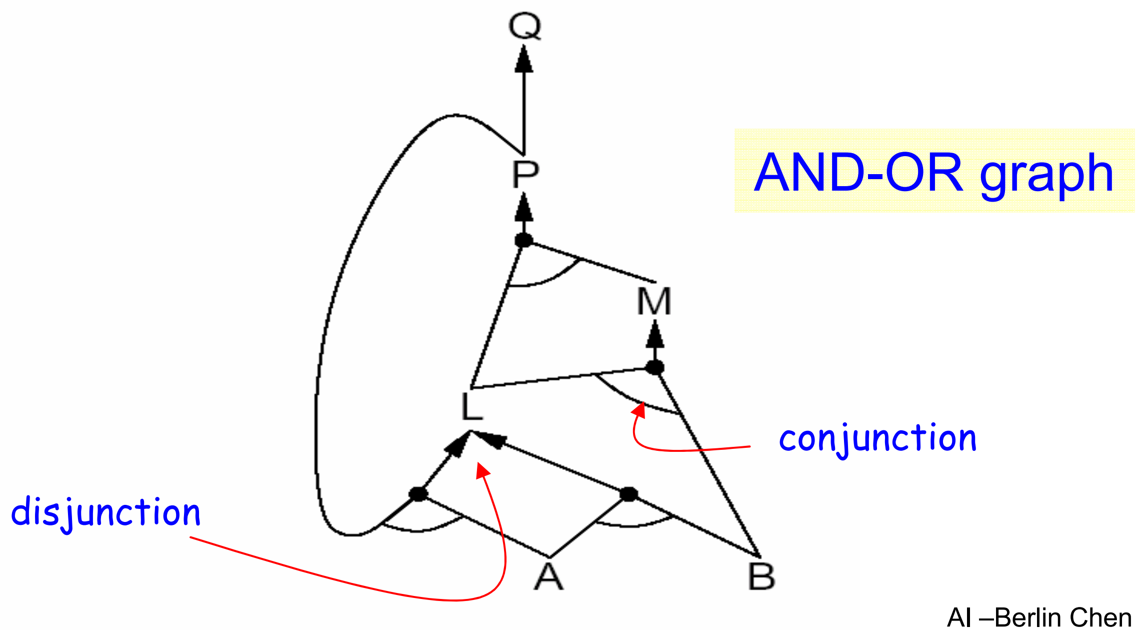
$$\frac{\alpha \Rightarrow \beta, \quad \alpha}{\beta}$$

- Not every *PL* sentence can be represented as a conjunction of Horn clauses

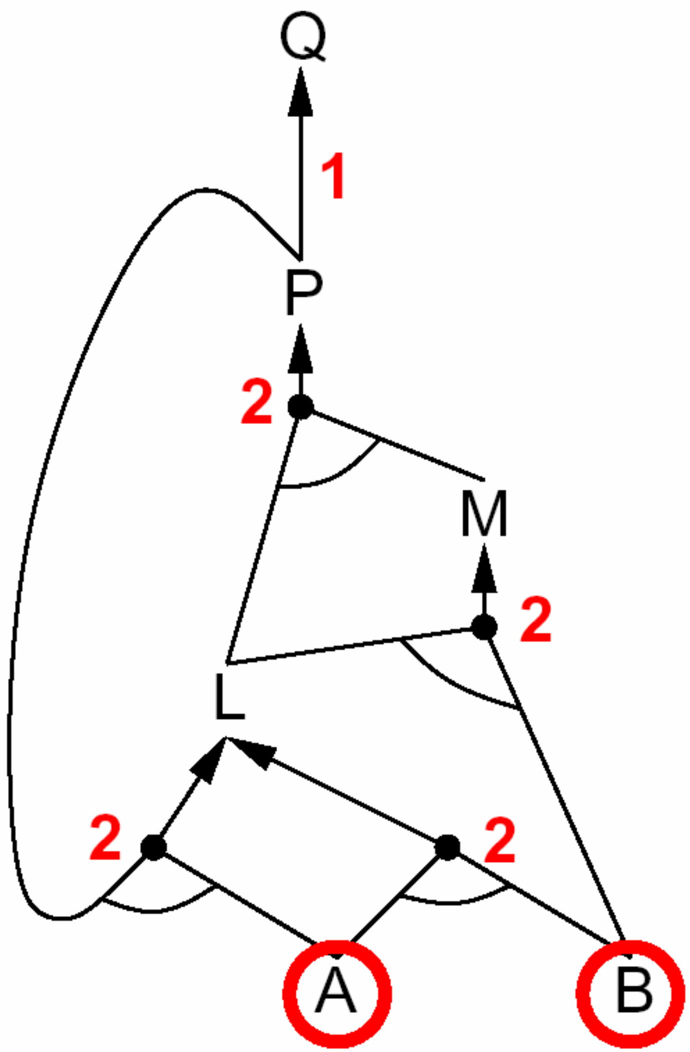
Forward Chaining

- As known, if all the premises of an implication are known, then its conclusion can be added to the set of known facts
- Forward Chaining fires any rule whose premises are satisfied in the *KB*, add its conclusion to the *KB*, until query is found or until no further inferences can be made
  - Applications of Modus Ponens

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

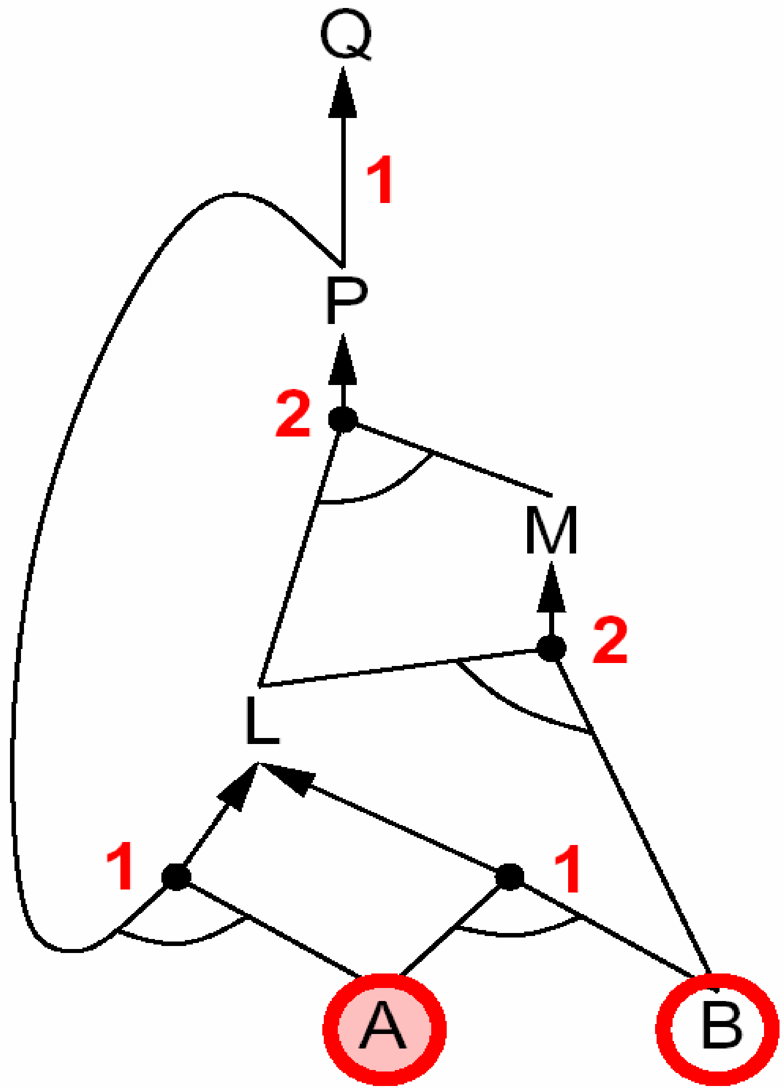


Forward Chaining: Example



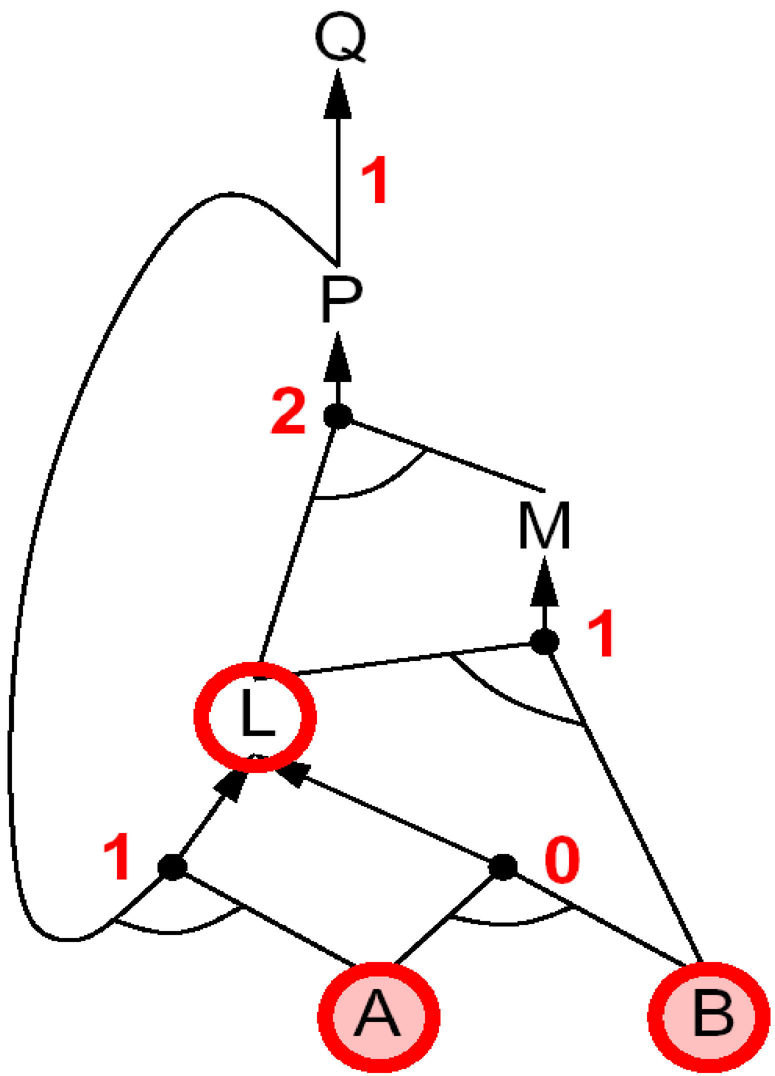
$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Forward Chaining: Example (cont.)



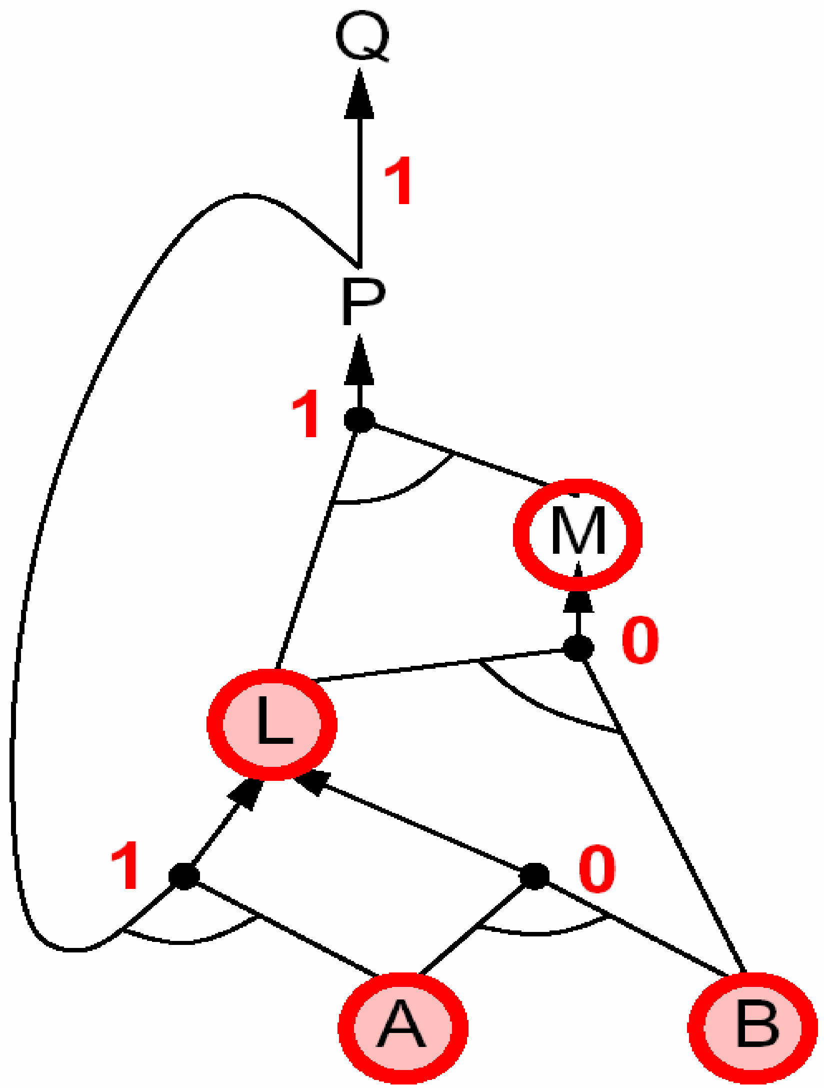
$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Forward Chaining: Example (cont.)



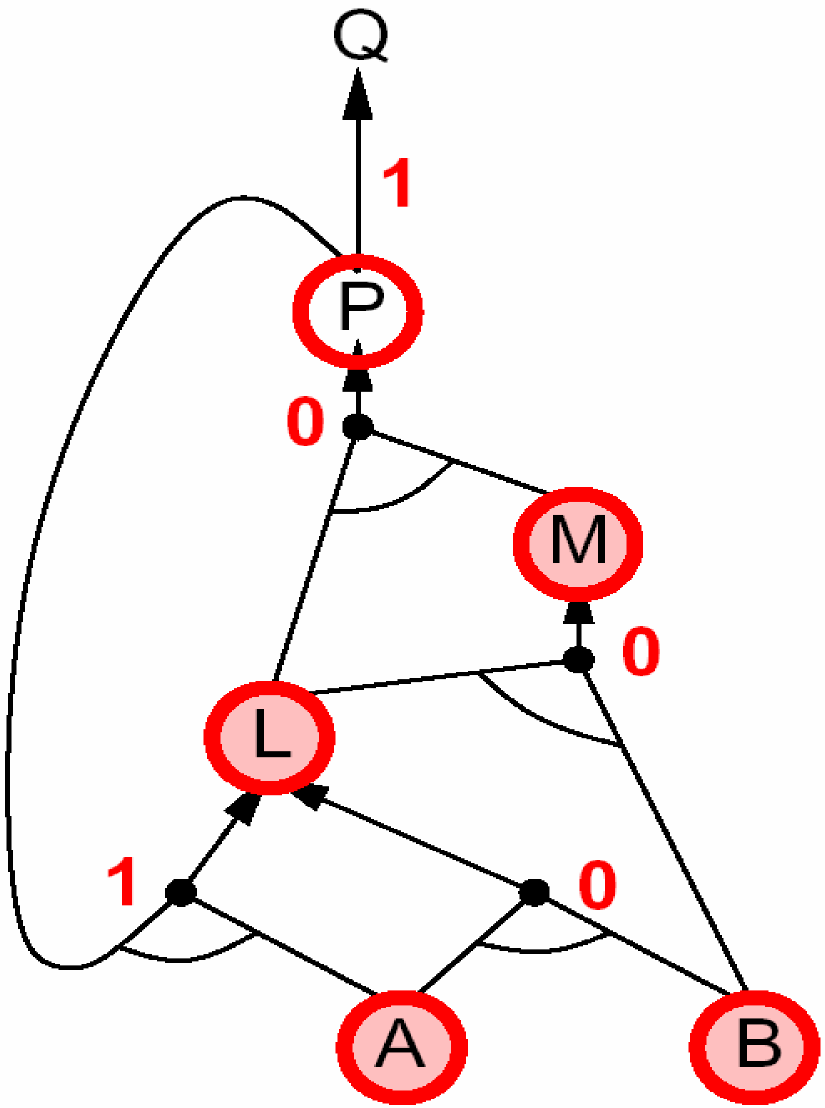
$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Forward Chaining: Example (cont.)



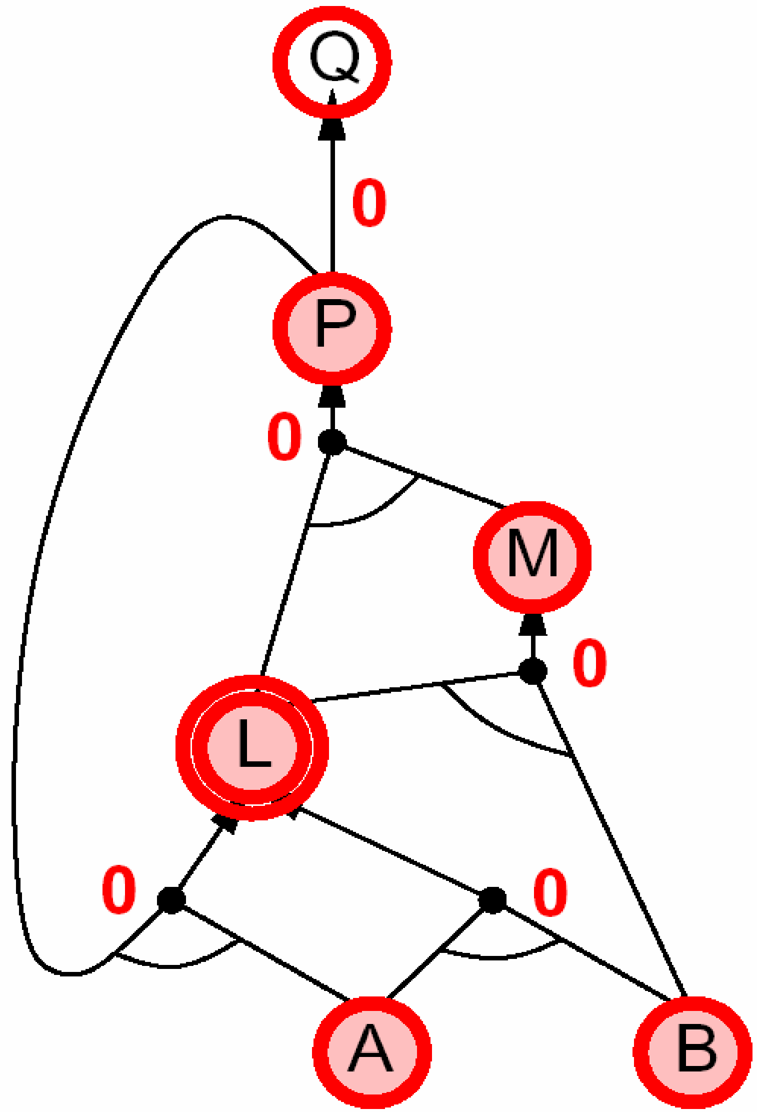
$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Forward Chaining: Example (cont.)



$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

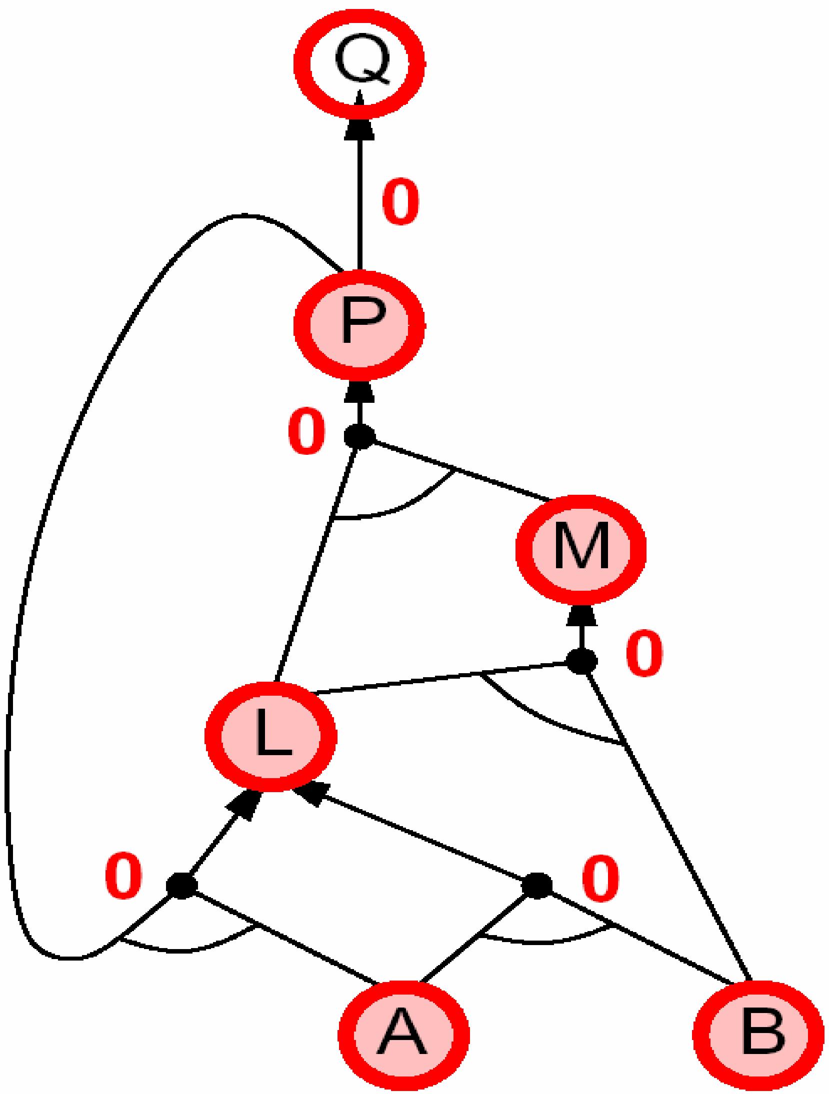
Forward Chaining: Example (cont.)



$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

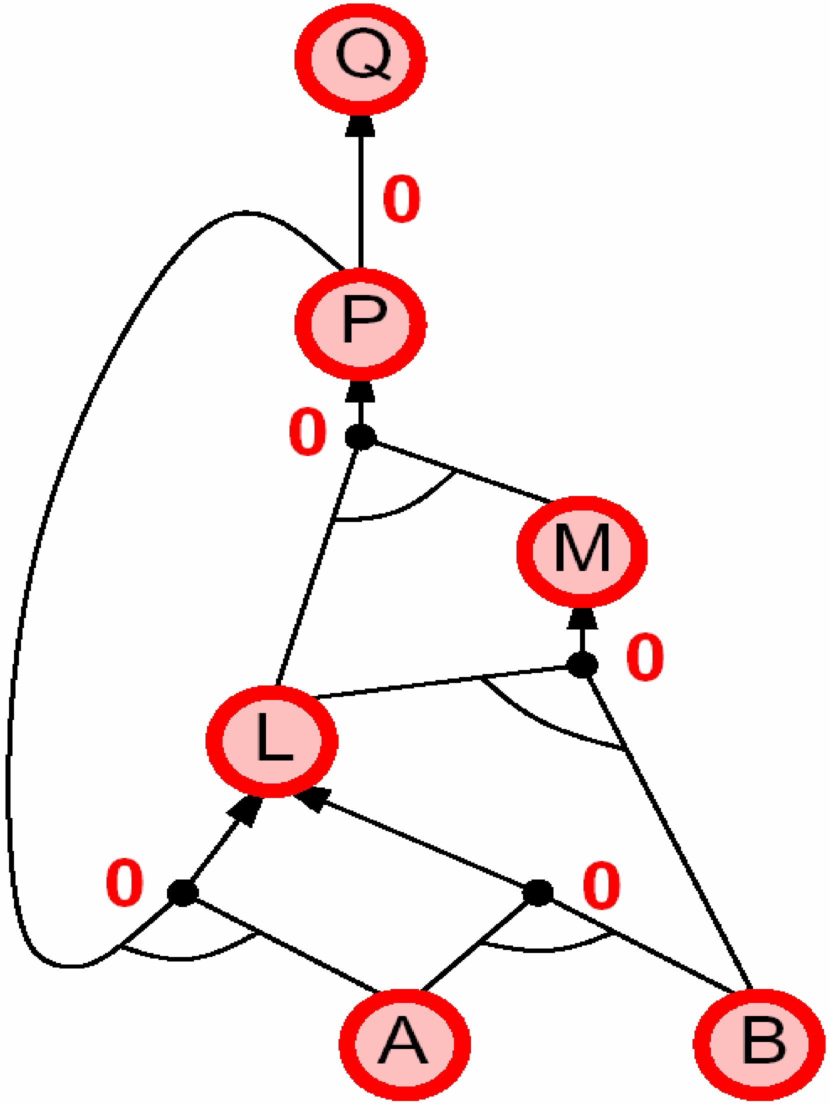
Firing *L*

Forward Chaining: Example (cont.)



$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Forward Chaining: Example (cont.)



$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Firing *Q*

Forward Chaining: Algorithm (cont.)

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  inputs: KB, the knowledge base, a set of propositional Horn clauses
          q, the query, a proposition symbol
  local variables: count, a table, indexed by clause, initially the number of premises
                   inferred, a table, indexed by symbol, each entry initially false
                   agenda, a list of symbols, initially the symbols known to be true in KB
                                                                              facts
  while agenda is not empty do
    p ← POP(agenda)
    unless inferred[p] do   //each logical symbol checked at most once (avoiding repeated firings)
      inferred[p] ← true
      for each Horn clause c in whose premise p appears do
        decrement count[c]
        if count[c] = 0 then do
          if HEAD[c] = q then return true
          PUSH(HEAD[c], agenda)
  return false
```

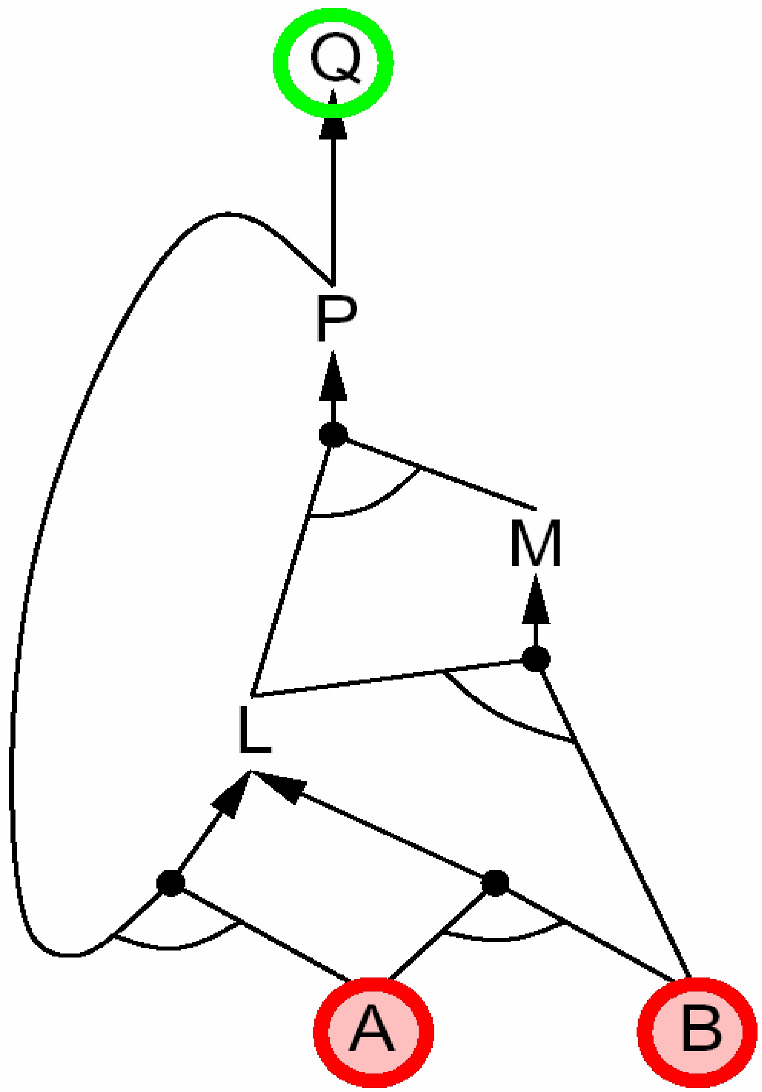
Forward Chaining: Properties

- Sound
  - Because every inference is an application of Modus Ponens
- Complete
  - Every entailed atomic sentence (i.e., propositional symbol) will be derived
  - But may do lots of work that is irrelevant to the goal
- A form of data-driven reasoning
  - Start with known data and derive conclusions from incoming percepts

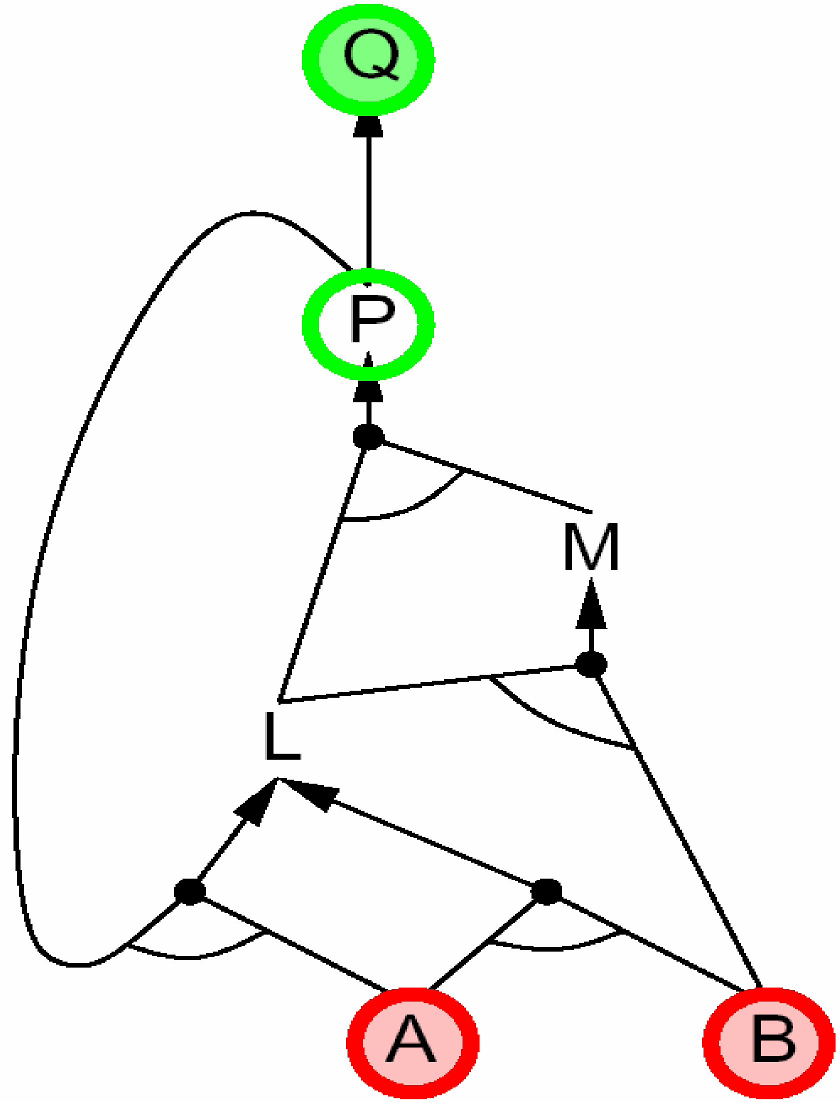
Backward Chaining

- Work backwards from the query *q* to prove *q* by backward chaining (*BC*)
- Check if *q* is known already, or prove by BC all premises of some rule concluding *q*
- A form of goal-directed reasoning

Backward Chaining: Example



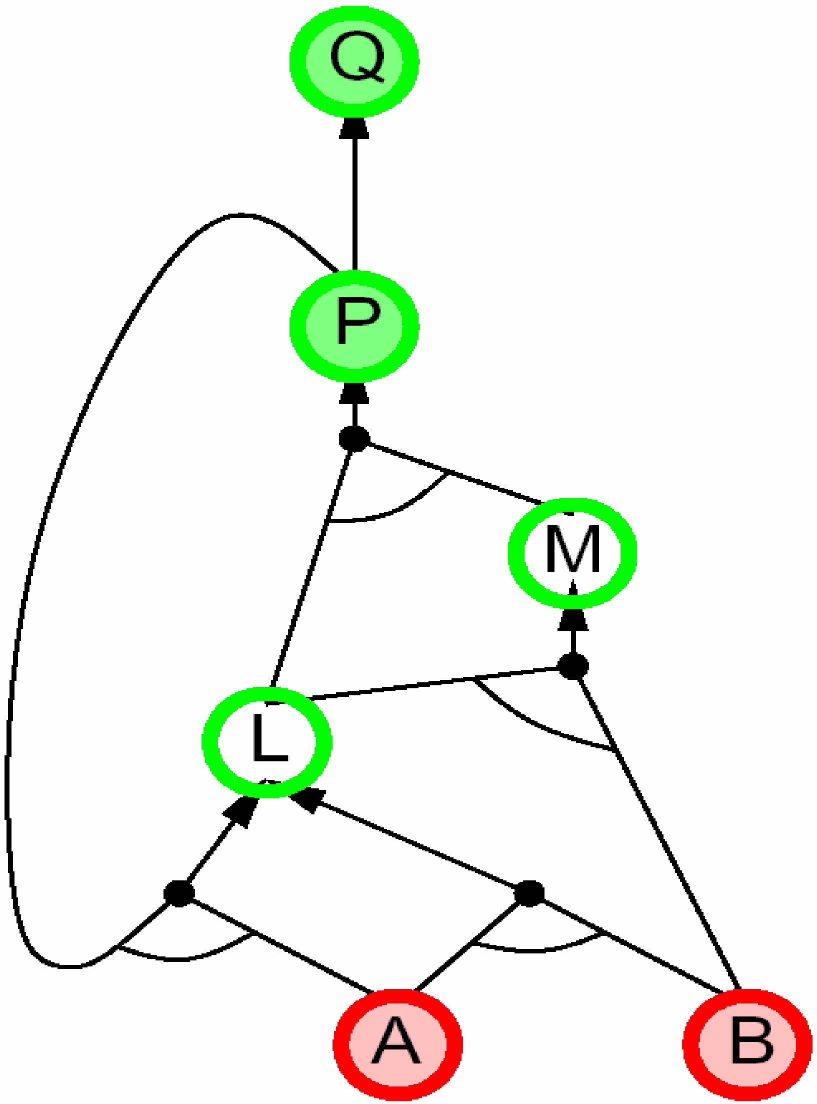
Backward Chaining: Example (cont.)



$P \Rightarrow Q$

$$
\begin{aligned}
P &\Rightarrow Q \\
L \wedge M &\Rightarrow P \\
B \wedge L &\Rightarrow M \\
A \wedge P &\Rightarrow L \\
A \wedge B &\Rightarrow L \\
A & \\
B &
\end{aligned}
$$

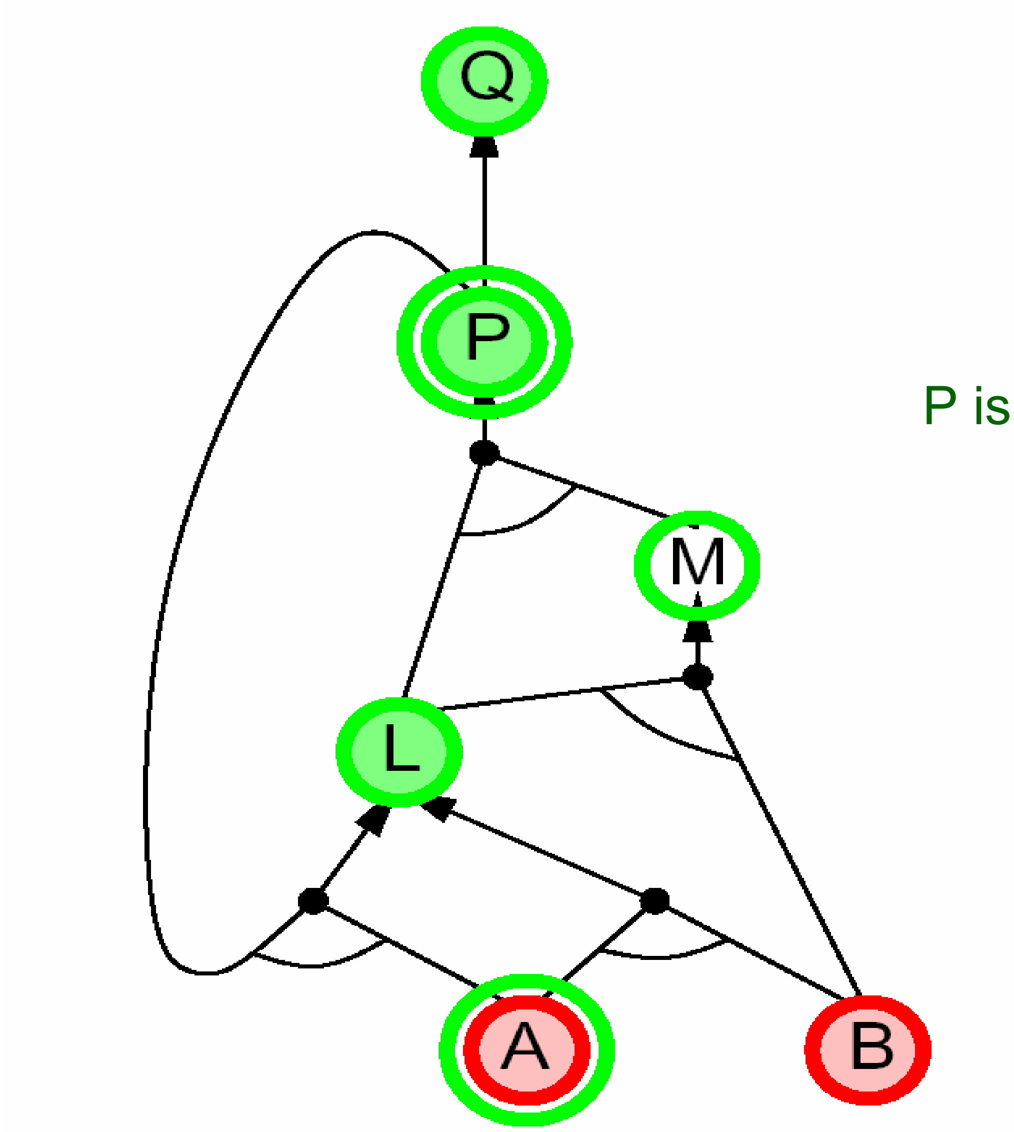
Backward Chaining: Example (cont.)



$L \wedge M \Rightarrow P$

$$
\begin{aligned}
P &\Rightarrow Q \\
L \wedge M &\Rightarrow P \\
B \wedge L &\Rightarrow M \\
A \wedge P &\Rightarrow L \\
A \wedge B &\Rightarrow L \\
A & \\
B &
\end{aligned}
$$

Backward Chaining: Example (cont.)

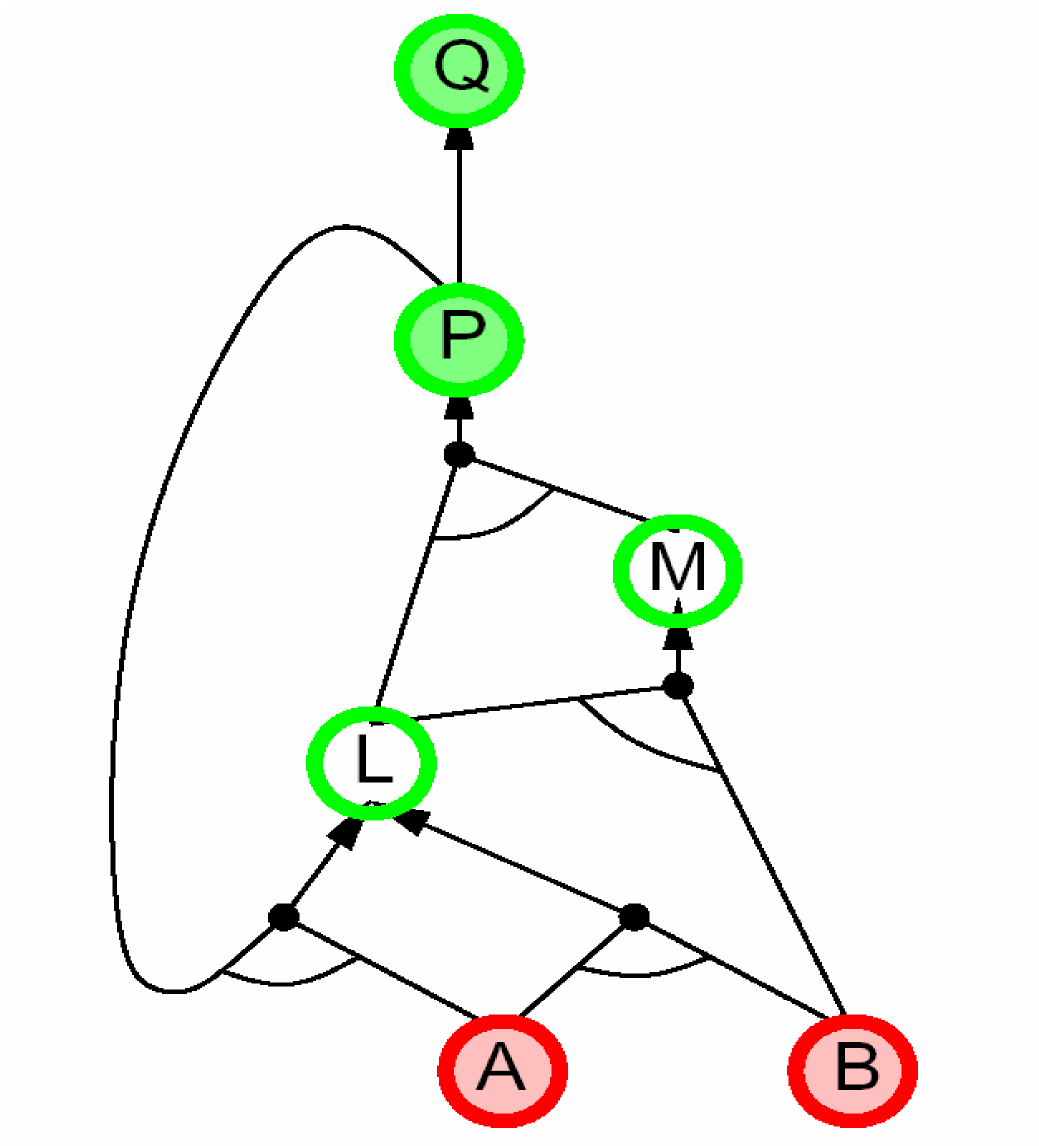


$A \wedge P \Rightarrow L$

P is not known already !

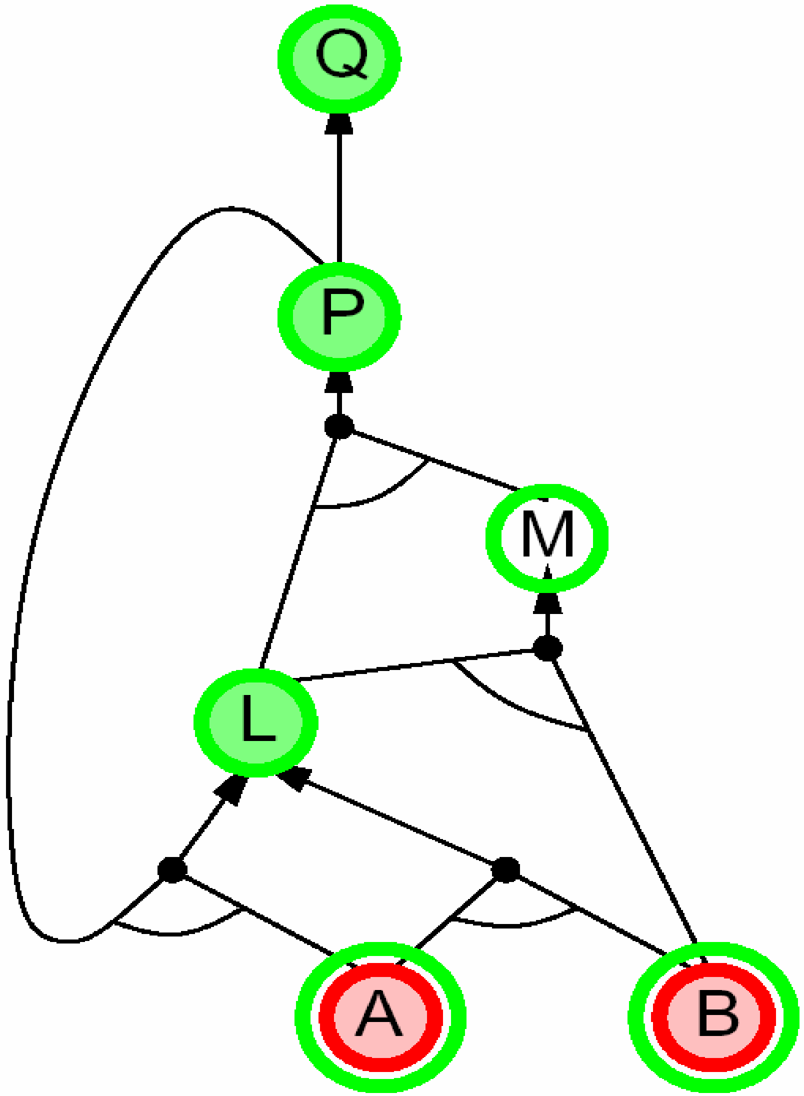
$$\begin{aligned} P &\Rightarrow Q \\ L \wedge M &\Rightarrow P \\ B \wedge L &\Rightarrow M \\ A \wedge P &\Rightarrow L \\ A \wedge B &\Rightarrow L \\ A & \\ B & \end{aligned}$$

Backward Chaining: Example (cont.)



$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Backward Chaining: Example (cont.)



$A \wedge B \Rightarrow L$

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
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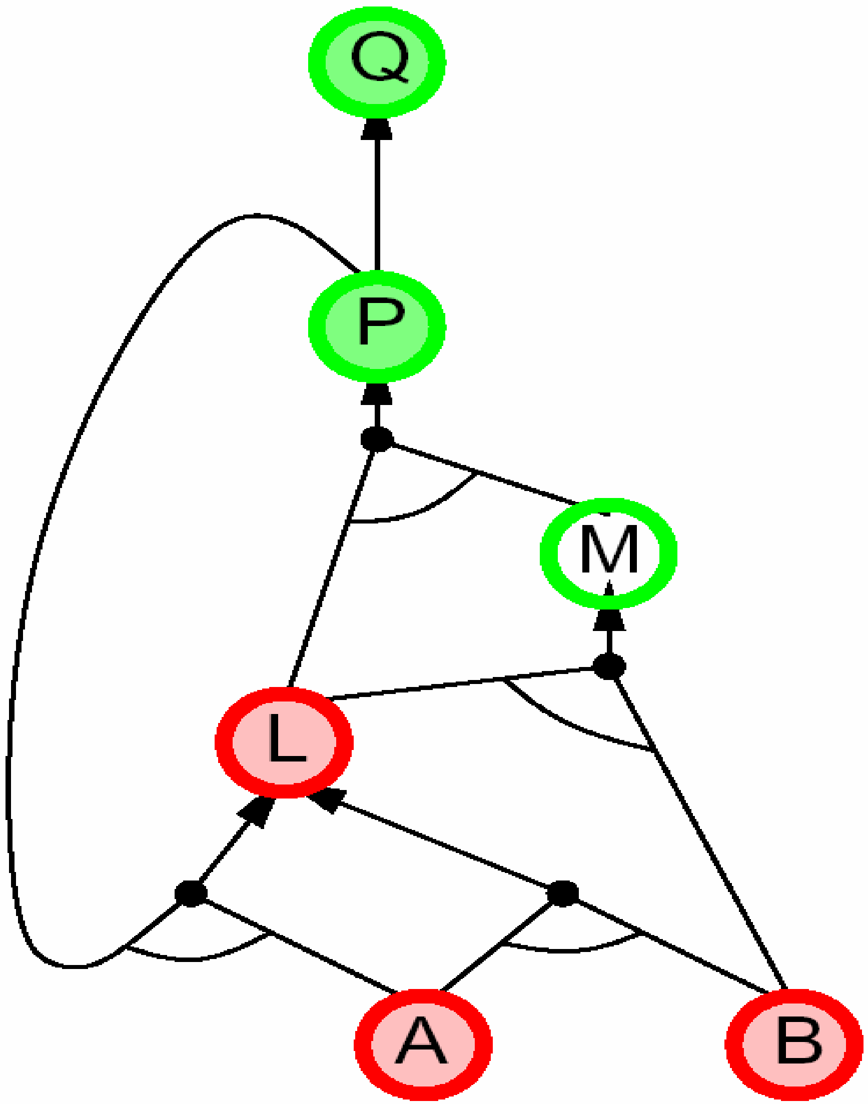
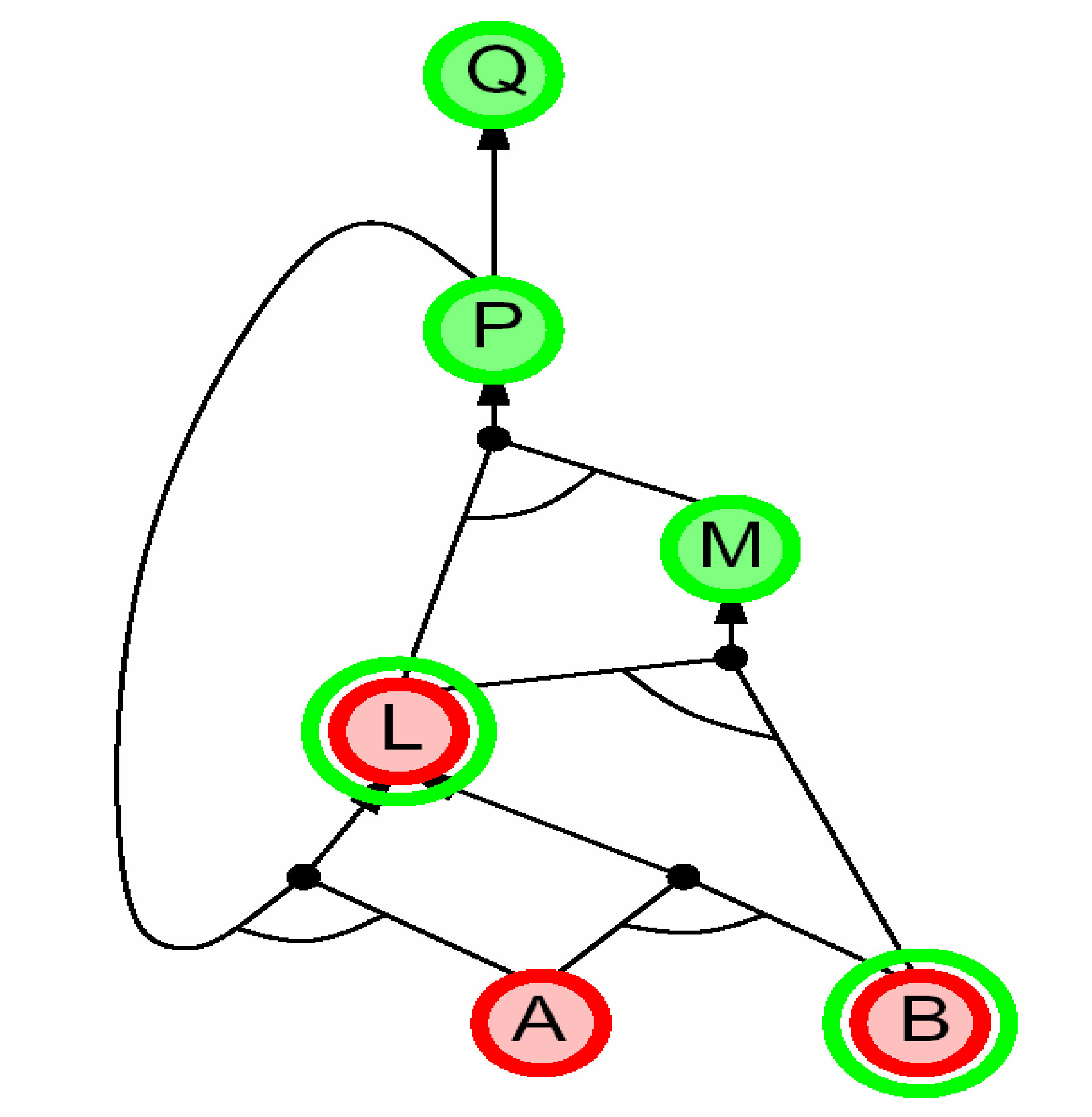
Backward Chaining: Example (cont.)



Backward Chaining: Example (cont.)



$B \wedge L \Rightarrow M$

$P \Rightarrow Q$

$L \wedge M \Rightarrow P$

$B \wedge L \Rightarrow M$

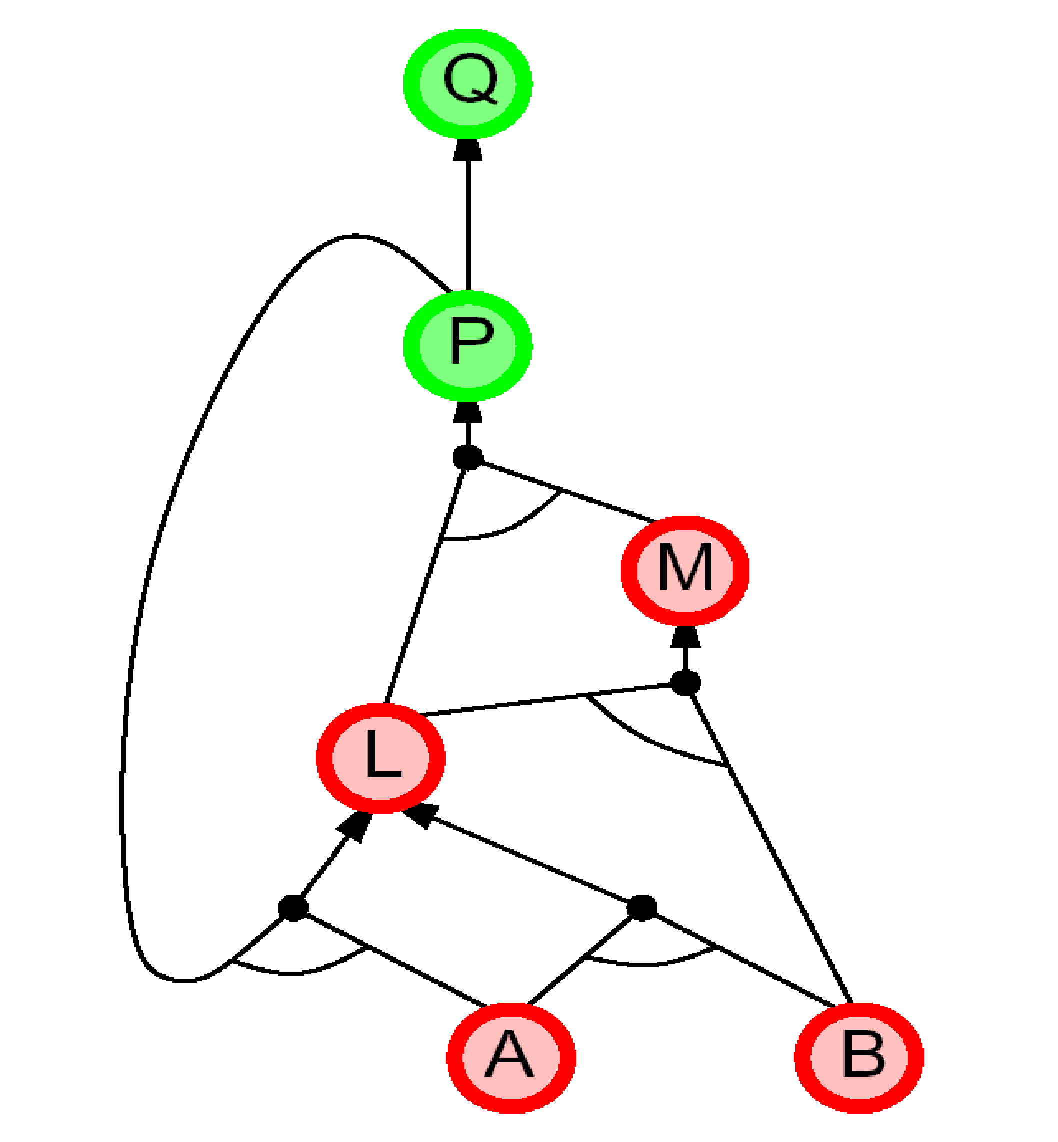
$A \wedge P \Rightarrow L$

$A \wedge B \Rightarrow L$

A

B

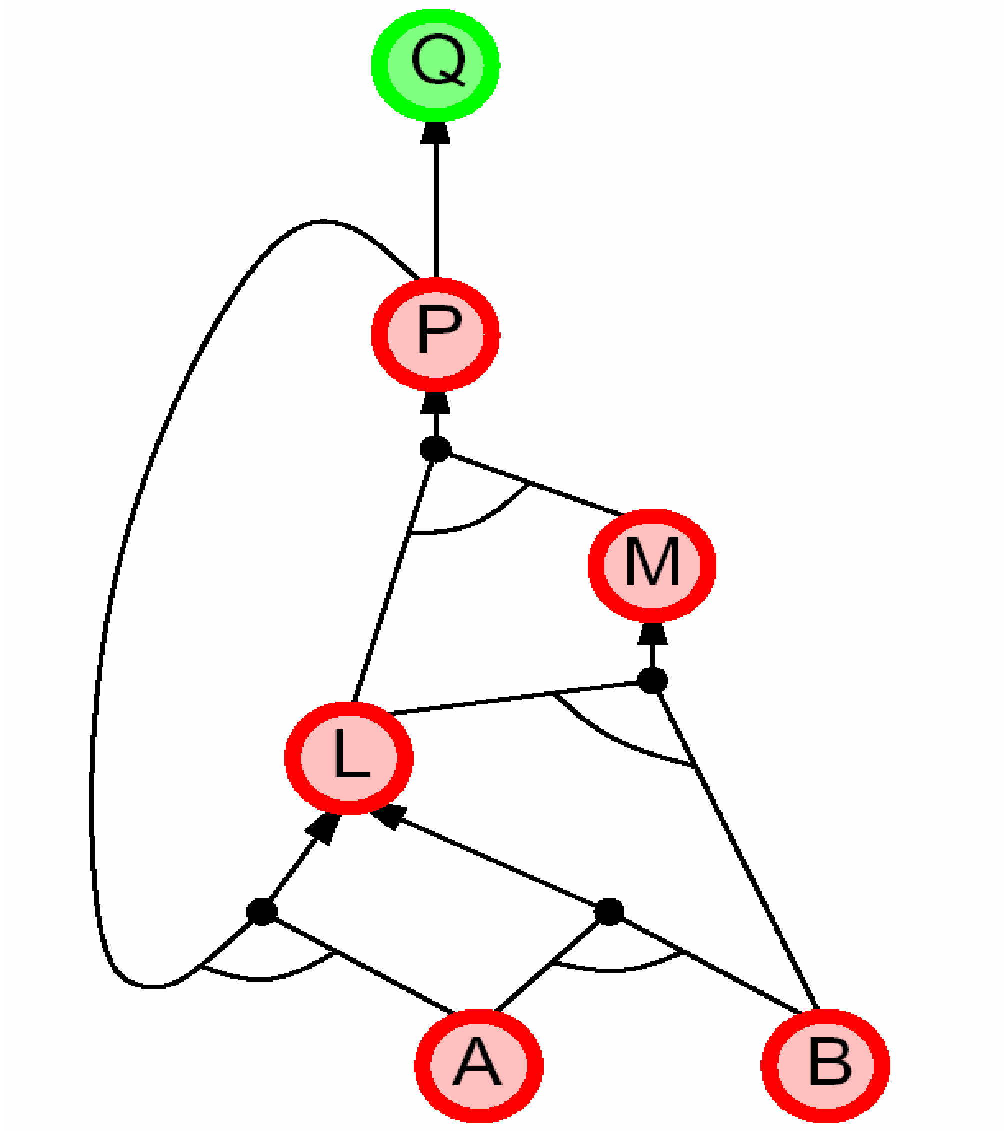
Backward Chaining: Example (cont.)



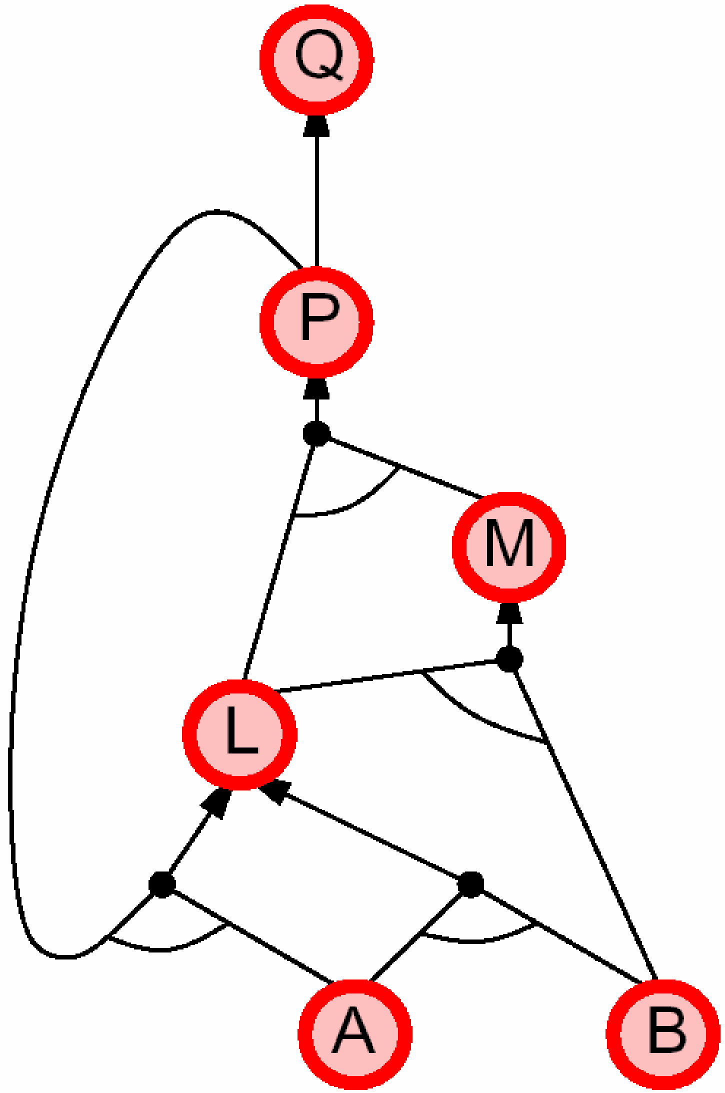
$L \wedge M \Rightarrow P$

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

Backward Chaining: Example (cont.)



Backward Chaining: Example (cont.)



$P \Rightarrow Q$

$$
\begin{aligned}
P &\Rightarrow Q \\
L \wedge M &\Rightarrow P \\
B \wedge L &\Rightarrow M \\
A \wedge P &\Rightarrow L \\
A \wedge B &\Rightarrow L \\
A & \\
B &
\end{aligned}
$$

Forward vs. Backward Chaining

- FC (data-driven)
  - May do lots of work that is irrelevant to the goal
- BC (goal-driven)
  - Complexity of BC can be much less than linear in size of *KB*

Propositional Logic: Drawbacks

- Propositional Logic is declarative and compositional
- The lack of expressive power to describe an environment with many objects concisely
  - E.g., we have to write a separate rule about breezes and pits for each square

$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$