Further Topics on Random Variables:
Conditional Expectation and Variance Revisited

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Reference:

- D. P. Bertsekas, J. N. Tsitsiklis, *Introduction to Probability* , Section 4.3

Revisit: Conditional Expectation and Variance

- Goal: To introduce two useful probability laws
  - **Law of Iterated Expectations**

$$\mathbf{E}\left[\mathbf{E}\left[X \mid Y\right]\right] = \mathbf{E}\left[X\right]$$

  - **Law of Total Variance**

$$\mathrm{var}(X) = \mathbf{E}\left[\mathrm{var}(X \mid Y)\right] + \mathrm{var}\left(\mathbf{E}\left[X \mid Y\right]\right)$$

More on Conditional Expectation

- Recall that the conditional expectation $\mathbf{E}[X|Y=y]$ is defined by

$$\mathbf{E}[X|Y=y]=\sum_x x\cdot p_{X|Y}(x|y), \qquad \text{(If } X \text{ is discrete)}$$

and

$$\mathbf{E}[X|Y=y]=\int_{-\infty}^{\infty} x\cdot f_{X|Y}(x|y)dx. \quad \text{(If } X \text{ is continuous)}$$

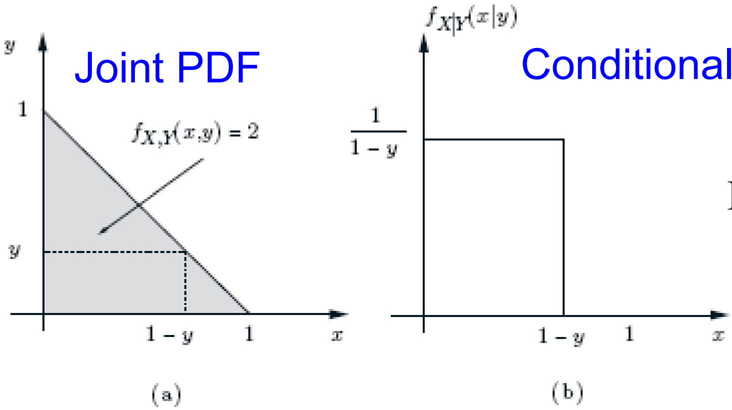
- $\mathbf{E}[X|Y=y]$ in fact can be viewed as a function of Y, because its value depends on the value y of Y
  - Is $\mathbf{E}[X|Y]$ a random variable ?
  - What is the expected value of $\mathbf{E}[X|Y]$?
    - Note also that the expectation of a function $g(Y)$ of Y

$$\mathbf{E}[g(Y)]=\begin{cases}\sum_y g(y)p_Y(y), & \text{if } Y \text{ is discrete} \\ \int_{-\infty}^{\infty} g(y)f_Y(y)dy, & \text{if } Y \text{ is continuous}\end{cases}$$

An Illustrative Example (1/2)

- **Example.** Let the random variables X and Y have a joint PDF which is equal to 2 for (x, y) belonging to the triangle indicated below and zero evervwhere else.

$$X + Y \le 1$$
$$0 \le X \le 1$$
$$0 \le Y \le 1$$



$$\mathbf{E}[X|Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)dx.$$

- What's the value of $\mathbf{E}[X|Y = y]$?

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$
$$= \int_0^{1-y} f_{X,Y}(x, y)dx \quad (\because X + Y \le 1)$$
$$= \int_0^{1-y} 2dx = 2(1 - y), \qquad 0 \le y \le 1$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{1}{1-y}$$

for $0 \le x \le 1 - y$ where $0 \le y \le 1$

$$\therefore \mathbf{E}[X|Y = y] = \int_0^{1-y} x \cdot \frac{1}{1-y} dx$$
$$= \frac{1}{2(1-y)} \cdot x^2 \Big|_0^{1-y} = \frac{1-y}{2}$$

(a linear function of Y)

An Illustrative Example (2/2)

– We saw that $\mathbf{E}[X|Y=y]=(1-y)/2$. Hence, $\mathbf{E}[X|Y]$ is the random variable $(1-Y)/2$:

$$\mathbf{E}[X|Y]=\frac{(1-Y)}{2}$$

– The expectation of $\mathbf{E}[X|Y]$

$$\mathbf{E}[\mathbf{E}[X|Y]]=\int_{-\infty}^{\infty}\mathbf{E}[X|Y=y]f_Y(y)dy=\mathbf{E}[X]$$

Total Expectation Theorem

For this problem, we thus have

$$\mathbf{E}[X]=\mathbf{E}[\mathbf{E}[X|Y]]=\mathbf{E}[(1-Y)/2]=(1-\mathbf{E}[Y])/2$$

$$\begin{aligned}\mathbf{E}[Y]&=\int_0^1 y\cdot f_Y(y)dy\\&=\int_0^1 y\cdot 2(1-y)dy\\&=y^2-(2/3)y^3\Big|_0^1\\&=1/3\end{aligned}$$

$$\therefore\ \mathbf{E}[X]=(1-\mathbf{E}[Y])/2=1/3$$

Law of Iterated Expectations

$$\mathbf{E}\left[\mathbf{E}\left[X \mid Y\right]\right] = \mathbf{E}\left[X\right]$$

$$\mathbf{E}\left[\mathbf{E}\left[X \mid Y\right]\right] = \begin{cases} \sum_{y} \mathbf{E}\left[X \mid Y = y\right] p_Y(y), & \text{(If } Y \text{ is discrete)} \\ \int_{-\infty}^{\infty} \mathbf{E}\left[X \mid Y = y\right] f_Y(y)\, dy. & \text{(If } Y \text{ is continuous)} \end{cases}$$

An Illustrative Example (1/2)

- **Example 4.17.** We start with a stick of length l. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process on the stick that we were left with.
  - What is the expected length of the stick that we are left with, after breaking twice?

Let Y be the length of the stick after we break for the first time.

Let X be the length after the second time.

$$f_Y(y) = \begin{cases} \frac{1}{l}, & \text{for } 0 \le y \le l \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & \text{for } 0 \le x \le y \\ 0, & \text{otherwise} \end{cases}$$

uniformly distributed

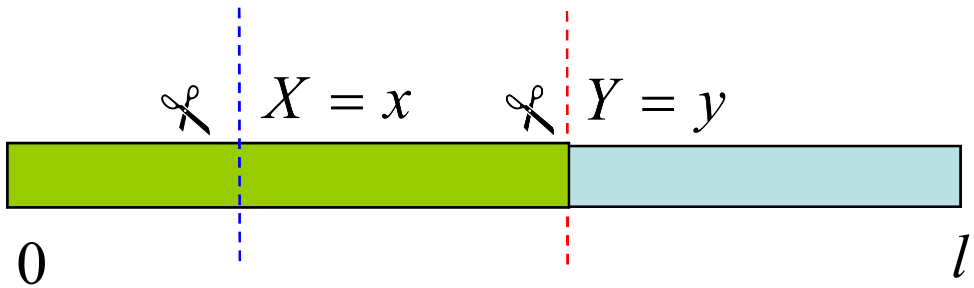
uniformly distributed

An Illustrative Example (2/2)

– By the **Law of Iterated Expectations**, we have

$$
\begin{aligned}
\mathbf{E}[X] &= \mathbf{E}\big[\mathbf{E}[X|Y]\big] \\
&= \int_0^l \mathbf{E}[X|Y=y] f_y(y)dy = \int_0^l \left[\int_0^y x f_{X|Y}(x|y)dx\right] f_y(y)dy \\
&= \int_0^l \left[\int_0^y x \frac{1}{y} dx\right] \tfrac{1}{l} dy = \int_0^l \left[\frac{1}{y} \cdot \frac{x^2}{2}\Big|_0^y\right] \tfrac{1}{l} dy \\
&= \tfrac{1}{l} \cdot \int_0^l \frac{y}{2} dy = \tfrac{1}{l} \cdot \frac{y^2}{4}\Big|_0^l \\
&= \frac{l}{4}
\end{aligned}
$$

Note that $\mathbf{E}[X|Y=y] = \frac{y}{2}$



Averaging by Section (1/3)

- Averaging by section can be viewed as a special case of the law of iterated expectations

- **Example 4.18. Averaging Quiz Scores by Section.**
  - A class has n students and the quiz score of student i is x_i. The average quiz score is

$$m = \frac{1}{n} \sum_{i=1}^{n} x_i$$

  - If students are divided into k disjoint subsets $A_1, A_2, \ldots, A_k$, the average score in section s is

$$m_s = \frac{1}{n_s} \sum_{x_i \in A_s} x_i$$

Averaging by Section (2/3)

- **Example 4.18. (cont.)**
  - The average score of over the whole class can be computed by taking a weighted average of the average score m_s of each class s , while the weight given to section s is proportional to the number of students in that section

$$
\begin{aligned}
\sum_{s=1}^{k} \frac{n_s}{n} m_s &= \sum_{s=1}^{k} \frac{n_s}{n} \cdot \frac{1}{n_s} \sum_{x_i \in A_s} x_i \\
&= \frac{1}{n} \underbrace{\sum_{s=1}^{k} \sum_{x_i \in A_s}}_{} x_i \\
&= \frac{1}{n} \underbrace{\sum_{i=1}^{n}}_{} x_i \\
&= m
\end{aligned}
$$

Averaging by Section (3/3)

- **Example 4.18. (cont.)**
  - **Its relationship with the law of iterated expectations**
    - Two random variable defined
      - X : quiz score of a student (or outcome)
        - » Each student (or outcome) is uniformly distributed
      - Y : section of a student $Y \in \{1, \dots, k\}$

$$\Rightarrow \mathbf{E}[X] = m \qquad (?)$$

$$\mathbf{E}[X \mid Y = s] = \frac{1}{n_s} \sum_{i \in A_s} x_i = m_s \qquad (?)$$

$$\because P(Y = s) = \frac{n_s}{n} \qquad (?)$$

$$\therefore m = \mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]] = \sum_{s=1}^{k} \mathbf{E}[X \mid Y = s] P(Y = s)$$

$$= \sum_{s=1}^{k} m_s \cdot \frac{n_s}{n}$$

More on Conditional Variance

- Recall that the conditional variance of X , given $Y=y$, is defined by

$$\mathrm{var}(X|Y=y)=\mathbf{E}\left[\left(X-\mathbf{E}[X|Y=y]\right)^2|Y=y\right]$$

- $\mathrm{var}(X|Y)$ in fact can be viewed as a function of Y , because its value $\mathrm{var}(X|Y=y)$ depends on the value y of Y
  - Is $\mathrm{var}(X|Y)$ a random variable ?
  - What is the expected value of $\mathrm{var}(X|Y)$?

Note that $\mathbf{E}[\mathrm{var}(X|Y)]\neq \mathrm{var}(X)$

Law of Total Variance

- The expectation of the conditional variance $\mathrm{var}(X|Y)$ is related to the unconditional variance $\mathrm{var}(X)$

$$\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbf{E}[X|Y])$$

$$\begin{aligned}
\mathrm{var}(X) &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\
&= \mathbf{E}[\mathbf{E}[X^2|Y]] - (\mathbf{E}[\mathbf{E}[X|Y]])^2 \\
&= \mathbf{E}\left[\mathrm{var}(X|Y) + (\mathbf{E}[X|Y])^2\right] - (\mathbf{E}[\mathbf{E}[X|Y]])^2 \\
&= \mathbf{E}[\mathrm{var}(X|Y)] + \mathbf{E}\left[(\mathbf{E}[X|Y])^2\right] - (\mathbf{E}[\mathbf{E}[X|Y]])^2 \\
&= \mathbf{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbf{E}[X|Y])
\end{aligned}$$

Law of Iterated Expectations

$$\mathbf{E}[X^2] = \mathbf{E}[\mathbf{E}[X^2|Y]]$$
$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]]$$

$$\mathrm{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2$$
$$\Rightarrow \mathbf{E}[Z^2] = \mathrm{var}(Z) + (\mathbf{E}[Z])^2$$

Illustrative Examples (1/4)

- **Example 4.17. (continued)** Consider again the problem where we break twice a stick of length , at randomly chosen points, with Y being the length of the stick after the first break and X being the length after the second break
  - Calculate $\text{var}(X)$ using the law of total variance

$$\text{var}(X) = \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y])$$

$$\text{We know that } f_Y(y) = \begin{cases} \frac{1}{l}, & \text{for } 0 \le y \le l \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & \text{for } 0 \le x \le y \\ 0, & \text{otherwise} \end{cases}$$

uniformly distributed — uniformly distributed

We also know that if a random variable Z is uniformly distributed in $[a, b]$, then its variance is

$$\text{var}(Z) = \frac{(b-a)^2}{12}$$

$$\Rightarrow \text{var}(X|Y=y) = \frac{y^2}{12}$$

$\Rightarrow$ "random variable" $\text{var}(X|Y)$ can be expressed as $\frac{Y^2}{12}$ (That is $\text{var}(X|Y) = \frac{Y^2}{12}$)

(a function of Y)

Illustrative Examples (2/4)

$$\mathbf{E}\left[\mathrm{var}(X|Y)\right] = \int_0^l \mathrm{var}(X|Y=y) f_Y(y)\,dy$$

$$= \int_0^l \frac{y^2}{12}\frac{1}{l}\,dy$$

$$= \frac{y^3}{36\cdot l}\Big|_0^l = \frac{l^2}{36}$$

$$\therefore\ \mathrm{var}(X) = \mathbf{E}\left[\mathrm{var}(X|Y)\right] + \mathrm{var}\left(\mathbf{E}[X|Y]\right)$$

$$= \frac{l^2}{36} + \frac{l^2}{48} = \frac{7\cdot l^2}{144}$$

Note that $\mathbf{E}[X|Y=y] = \dfrac{y}{2}$ cf. p.14

$$\Rightarrow \mathbf{E}[X|Y] = \frac{Y}{2}$$ (a function of Y)

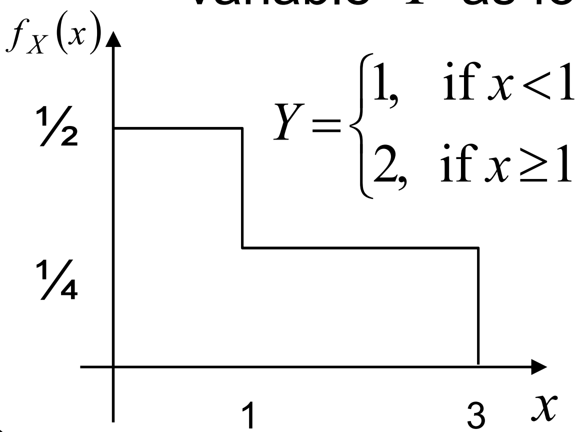
$$\Rightarrow \mathrm{var}\left(\mathbf{E}[X|Y]\right)$$

$$= \mathrm{var}\left(\frac{Y}{2}\right) = \frac{1}{4}\mathrm{var}(Y)$$

$$= \frac{1}{4}\cdot\frac{l^2}{12} = \frac{l^2}{48}$$ (Y is uniformly distributed)

Illustrative Examples (3/4)

- **Example 4.21.** Computing Variances by Conditioning.
  - Consider a continuous random variable X with the PDF given in the following figure. We define an auxiliary (discrete) random variable Y as follows:



$$Y = \begin{cases} 1, & \text{if } x < 1 \\ 2, & \text{if } x \geq 1 \end{cases}$$

$$\operatorname{var}(X) = \mathbf{E}\left[\operatorname{var}(X \mid Y)\right] + \operatorname{var}\left(\mathbf{E}[X \mid Y]\right)$$

We know that if a random variable Z is uniformly distributed in $[a, b]$, then its variance is

$$\operatorname{var}(Z) = \frac{(b-a)^2}{12}$$

①

$$\Rightarrow p_Y(1) = \int_0^1 1/2\,dx = 1/2$$

$$p_Y(2) = \int_1^3 1/4\,dx = 1/2$$

$$\Rightarrow f_{X|Y}(x \mid Y=1) = \begin{cases} 1, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x \mid Y=2) = \begin{cases} 1/2, & \text{for } 1 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

②

$$\operatorname{var}(X \mid Y=1) = (1-0)^2/12 = 1/12$$

$$\operatorname{var}(X \mid Y=2) = (3-1)^2/12 = 1/3$$

$$\Rightarrow \mathbf{E}\left[\operatorname{var}(X \mid Y)\right] = \operatorname{var}(X \mid Y=1)\,p_Y(1) + \operatorname{var}(X \mid Y=2)\,p_Y(2)$$
$$= 1/12 \cdot 1/2 + 1/3 \cdot 1/2 = 5/24$$

Note that $f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in A)}, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$

Illustrative Examples (4/4)

We know that if a random variable Z is uniformly distributed in $[a, b]$, then its mean is

$$E[Z] = \frac{a+b}{2}$$

③

$$\Rightarrow \mathbf{E}[X|Y=1] = (0+1)/2 = 1/2$$

$$\mathbf{E}[X|Y=2] = (1+3)/2 = 2$$

$$\mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}[X]$$

$$= \mathbf{E}[X|Y=1]p_Y(1) + \mathbf{E}[X|Y=2]p_Y(2)$$

$$= 1/2 \cdot 1/2 + 2 \cdot 1/2 = 5/4$$

$$\text{var}(\mathbf{E}[X|Y]) =$$

$$(\mathbf{E}[X|Y=1] - \mathbf{E}[\mathbf{E}[X|Y]])^2 p_Y(1)$$

$$+ (\mathbf{E}[X|Y=2] - \mathbf{E}[\mathbf{E}[X|Y]])^2 p_Y(2)$$

$$= (1/2 - 5/4)^2 \cdot 1/2 + (2 - 5/4)^2 \cdot 1/2$$

$$= 9/16$$

Note that for discrete random variable Z

$$\text{var}(Z) = \mathbf{E}\left[(Z - \mathbf{E}(Z))^2\right] = \sum_z (z - \mathbf{E}(Z))^2 p_Z(z)$$

④

$$\therefore \text{var}(X) = \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y])$$

$$= 9/16 + 5/24 = 37/48$$

Justification

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

$$= \int_0^1 x \cdot 1/2dx + \int_1^3 x \cdot 1/4dx$$

$$= \frac{1}{4}x^2\Big|_0^1 + \frac{1}{8}x^2\Big|_1^3$$

$$= 5/4$$

$$\text{var}(x) = \int_{-\infty}^{\infty} (x - \mathbf{E}[X])^2 \cdot f_X(x)dx$$

$$= \int_0^1 (x-5/4)^2 \cdot 1/2dx + \int_1^3 (x-5/4)^2 \cdot 1/4dx$$

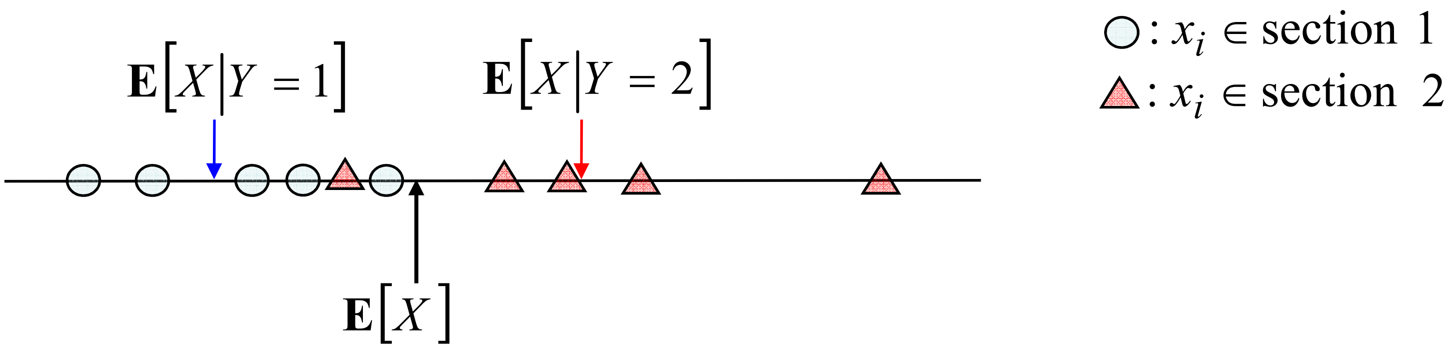
$$= \frac{1}{6}(x-5/4)^3\Big|_0^1 + \frac{1}{12}(x-5/4)^3\Big|_1^3$$

$$= \frac{1}{6}\left((-1/4)^3 - (-5/4)^3\right) + \frac{1}{12}\left((7/4)^3 - (-1/4)^3\right)$$

$$= \frac{1}{6} \cdot \frac{124}{64} + \frac{1}{12} \cdot \frac{344}{64} = \frac{37}{48}$$

Averaging by Section

- For a two-section (or two-cluster) problem



$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]] = \sum_s \mathbf{E}[X|Y=s]P(Y=s)$$

$$\text{var}(X) = \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y])$$

These two measures have been widely used for linear discriminant analysis (LDA)

average variability within individual sections

variability of $\mathbf{E}[X|Y]$ (the outcome means of individual sections)

Also called “within cluster” variation

Also called “between cluster” variation

Properties of Conditional Expectation and Variance

- $\mathbf{E}[X \mid Y = y]$ is a number, whose value depends on y.
- $\mathbf{E}[X \mid Y]$ is a function of the random variable Y, hence a random variable. Its experimental value is $\mathbf{E}[X \mid Y = y]$ whenever the experimental value of Y is y.
- $\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[X]$ (law of iterated expectations).
- $\text{var}(X \mid Y)$ is a random variable whose experimental value is $\text{var}(X \mid Y = y)$, whenever the experimental value of Y is y.
- $\text{var}(X) = \mathbf{E}\big[\text{var}(X \mid Y)\big] + \text{var}\big(\mathbf{E}[X \mid Y]\big)$.