Adversarial Search

References:
3. S. Russell’s teaching materials
Introduction

• Game theory
  – First developed by von Neumann and Morgenstern
  – Widely studied by economists, mathematicians, financiers, etc.
  – The action of one player (agent) can significantly affect the utilities of the others
    • Cooperative or competitive
    • Deal with the environments with multiple agents

• Most games studied in AI are
  – Deterministic (but strategic) \((\text{state, action(state)}) \rightarrow \text{next state}\)
  – Turn-taking
  – Two-player
  – Zero-sum
  – Perfect information

But not physical games

This means deterministic, fully observable environments in which there are two agents whose actions must alternate and in which the utility values at the end of game are always equal or opposite
Types of Games

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- Games are one of the first tasks undertaken in AI
  - The abstract nature of (nonphysical) games makes them an appealing subject in AI
- Computers have surpassed humans in checkers and Othello, and have defeated human champions in chess and backgammon
- However, in Go, computers still perform at the amateur level
Games as Search Problems

- Games are usually too hard to solve
  - E.g., a chess game
    - Average branching factor: 35
    - Average moves by each player: 50
    - Total number of nodes in the search tree: $35^{100}$ or $10^{154}$
    - Total number of distinct states: $10^{40}$

- The solution is a strategy that specifies a move for every possible opponent reply
  - Time limit: how to make the best possible use of time?
    - Calculate the optimal decision may be infeasible
    - Pruning is needed
  - Uncertainty: due to the opponent’s actions and game complexity
    - Imperfect information
    - Chance
Scenario

- Games with two players
  - MAX, moves first
  - MIN, moves second

- At the end of the game
  - Winner awarded and loser penalized
  - Or, draw

- Can be formally defined as a kind of search problem

\[ \text{Sense} \rightarrow \text{Plan} \rightarrow \text{Act} \]
Games as Search Problems

- Main components should be specified:
  - **Initial State**
    - Board position, which player to move
  - **Successor Function**
    - A list of legal \((move, state)\) pairs for each state indicating a legal move and the resulting state
  - **Terminal Test**
    - Determine when the game is over
    - Terminal states: states where the game has ended
  - **Utility Function** (objective/payoff function)
    - Give numeric values for all terminal states, e.g.:
      - Win, loss or draw : +1, -1, 0
      - Or values with a wider variety

Define the game tree

From the viewpoint of MAX
Example Game Tree for Tic-Tac-Toe

- Tic-Tac-Toe also called Noughts and Crosses
  - 2-player, deterministic, alternating

  - The numbers on leaves indicate the utility values of terminal states from the point of view of the MAX.
Minimax Search

• A strategy/solution for optimal decisions

• Examine the minimax value of each node in the game tree

\[ \text{Minmax– Value}(n) = \begin{cases} 
\text{Utility}(n) & \text{if } n \text{ is a terminal state} \\
\max_{s \in \text{Successor}(n)} \text{Minmax– Value}(s) & \text{if } n \text{ is a MAX node} \\
\min_{s \in \text{Successor}(n)} \text{Minmax– Value}(s) & \text{if } n \text{ is a MIN node}
\end{cases} \]

– Minmax value of a terminal state is just the utility from the point of view of MAX

– Assume two players (MAX and MIN) play optimally (infallibly) from the current node to the end of the game
Minimax Search (cont.)

• Example: a trivial 2-ply (one-move-deep) game
  – Perfect play for the deterministic, perfect-information game
    • MAX and MIN play optimally
  – Idea: Choose the move to a position with highest minimax value
    = best achievable payoff against best play
Tree for Tic-Tac-Toe

MAX's move

Start node

MAX

MIN

6 - 4 = 2
5 - 4 = 1
4 - 6 = -2
6 - 6 = 0
5 - 6 = -1
5 - 5 = 0
5 - 6 = -1
4 - 5 = -1
5 - 5 = 0
6 - 5 = 1
5 - 5 = 0
6 - 5 = 1
Tree for Tic-Tac-Toe (cont.)

MAX's move

MIN

Start node

MAX

MIN

Node 1

Node 0

Node 1

Node 0

Node 1

Node 0

Node 0

Node 0

Node 1

Node 0

Node 0

Node 0
Tree for Tic-Tac-Toe (cont.)
Minimax Search: Algorithm

function MINIMAX-DECISION(state) returns an action
    inputs: state, current state in game
    $v \leftarrow \text{MAX-VALUE(state)}$
    return the action in SUCCESSORS(state) with value $v$

function MAX-VALUE(state) returns a utility value
    if TERMINAL-TEST(state) then return UTILITY(state)
    $v \leftarrow -\infty$
    for $a, s$ in SUCCESSORS(state) do
        $v \leftarrow \text{MAX}(v, \text{MIN-VALUE}(s))$
    return $v$

function MIN-VALUE(state) returns a utility value
    if TERMINAL-TEST(state) then return UTILITY(state)
    $v \leftarrow \infty$
    for $a, s$ in SUCCESSORS(state) do
        $v \leftarrow \text{MIN}(v, \text{MAX-VALUE}(s))$
    return $v$
Minimax Search: Example

$v_A = -\infty$

$v_B = \infty$

$v_B = 3$

Terminal-Test

$v_B = 3$

Backed up to root
Minimax Search: Example (cont.)

The diagram shows a minimax search example with a decision tree. The values at each node are as follows:

1. The root node is A with a value of 3.
2. Node B has a value of 3, and node C has a value of ∞.
3. At the next level, node B has values 3 and 12, with node C having values 8 and 2.
4. At the next level, node B has values 3 and 12, with node C having values 8 and 2.
5. The algorithm backtracks to the root, adjusting values as needed.

The values are updated as the algorithm progresses, backtracking to ensure the optimal decision is made.
Minimax Search: Example (cont.)

\[ v_A = 3 \]

\[ v_B = 3 \]

\[ v_C = 2 \]

\[ v_D = \infty \]

\[ v_A = 3 \]

\[ v_B = 3 \]

\[ v_C = 2 \]

\[ v_D = 14 \]

\[ v_A = 3 \]

\[ v_B = 3 \]

\[ v_C = 2 \]

\[ v_D = 5 \]

\[ v_A = 3 \]

\[ v_B = 3 \]

\[ v_C = 2 \]

\[ v_D = 2 \]
Minimax Search: Example (cont.)

\[ v_A = 3 \]
\[ v_B = 3 \]
\[ v_C = 2 \]
\[ v_D = 2 \]

Backed up to root
Minimax Search (cont.)

- Explanations of the Minmax Algorithm
  - A complete depth-first, recursive exploration of the game tree
  - The utility function is applied to each terminal state
  - The utility (min or max values) of internal tree nodes are calculated and then backed up through the tree as the recursion unwind
  - At the root, MAX chooses the move leading to the highest utility
Properties of Minimax Search

• Is complete if tree is finite

• Is optimal if the opponent acts optimally

• Time complexity: $O(b^m)$
  – $m$: the maximum depth of the tree

• Space complexity: $O(bm)$ or $O(m)$ (when successors generated one at a time)

For chess, $b \approx 35$, $m \approx 100$ for “reasonable” games.
I.e., exact solution is completely infeasible.
Optimal Decisions in Multiplayer Games

- Extend the minimax idea to multiplayer games
- Replace the single value for each node with a vector of values (utility vector)

If A and B are in an alliance

- Alliances among players would be involved sometimes
  - E.g., A and B form an alliance to attack C
α - β Pruning

• The problem with minimax search
  – The number of nodes to examine is exponential in the number of moves

• α - β pruning
  – Applied to the minimax tree
  – Return the same moves as minimax would, but prune away branches that can’t possibly influence the final decision

• α : the value of best (highest-value) choice so far in search of MAX
• β : the value of best (lowest-value) choice so far in search of MIN
$\alpha - \beta$ Pruning (cont.)

- Example

The subtree to be explored (considered) next should have a utility equal to or higher than 3
\( \alpha - \beta \) Pruning (cont.)

- **Example**

The utility of this subtree will be no more than 2 (lower than current \( \alpha \)), so the remaining children can be pruned.
\( \alpha - \beta \) Pruning (cont.)

- Example

```
MAX

MIN

A ≥ 3

B 3

C ≤ 2

D ≤ 14

3 12 8

X

2

X

14
```
\( \alpha - \beta \) Pruning (cont.)

- Example

```
MAX

MIN

A \geq 3

B 3

C \leq 2

D \leq 5

X

X

3 12 8 2

14 5
```
\(\alpha - \beta\) Pruning (cont.)

- **Example**

  ![Diagram](image)

  Can't prune any successors of D at all because the worst successors of D have been generated first.
$\alpha - \beta$ Pruning (cont.)

(a) $[-\infty, +\infty]$ $\bigtriangleup$

$[-\infty, 3]$ $\bigtriangleup$

3

(b) $[-\infty, +\infty]$ $\bigtriangleup$

$[-\infty, 3]$ $\bigtriangleup$

3 12

(c) $[3, +\infty]$ $\bigtriangleup$

$[3, 3]$ $\bigtriangleup$

3 12 8

(d) $[3, +\infty]$ $\bigtriangleup$

$[3, 3]$ $\bigtriangleup$

3 12 8 2

(e) $[3, 14]$ $\bigtriangleup$

$[3, 3]$ $\bigtriangleup$

$[-\infty, 2]$ $\bigtriangleup$

$[-\infty, 14]$ $\bigtriangleup$

3 12 8 2 14

(f) $[3, 3]$ $\bigtriangleup$

$[3, 3]$ $\bigtriangleup$

$[-\infty, 2]$ $\bigtriangleup$

$[-\infty, 2]$ $\bigtriangleup$

$[2, 2]$ $\bigtriangleup$

3 12 8 2 14 5 2
\( \alpha - \beta \) Pruning (cont.)

Minmax Value(root) = \( \max(\min(3, 12, 8), \min(2, x, y), \min(14, 5, 2)) \)

= \( \max(3, \min(2, x, y), 2) \)

= \( \max(3, z, 2) \) \quad \text{where } z \leq 2

= 3

- The value of the root are independent of the value of the pruned leaves \( x \) and \( y \)
Tree for Tic-Tac-Toe (cont.)

Alpha value = -1

Start node

Beta value = -1

End node
If $m$ is better than $n$ for Player (MAX), $n$ will not be visited in play and can therefore be pruned.
**α - β** Pruning (cont.)

- **Algorithm**

  ```python
  function ALPHA-BETA-SEARCH(state) returns an action
  inputs: state, current state in game
  
v ← MAX-VALUE(state, −∞, +∞)
return the action in SUCCESSORS(state) with value v

  function MAX-VALUE(state, α, β) returns a utility value
  inputs: state, current state in game
  α, the value of the best alternative for MAX along the path to state
  β, the value of the best alternative for MIN along the path to state

  if TERMINAL-TEST(state) then return UTILITY(state)
v ← −∞
for a, s in SUCCESSORS(state) do
  v ← MAX(v, MIN-VALUE(s, α, β))
  if v ≥ β then return v
  α ← MAX(α, v)
return v

  function MIN-VALUE(state, α, β) returns a utility value
  inputs: state, current state in game
  α, the value of the best alternative for MAX along the path to state
  β, the value of the best alternative for MIN along the path to state

  if TERMINAL-TEST(state) then return UTILITY(state)
v ← +∞
for a, s in SUCCESSORS(state) do
  v ← MIN(v, MAX-VALUE(s, α, β))
  if v ≤ α then return v
  β ← MIN(β, v)
return v
  ```

- Pruning: If one of its children has value larger than that of its best MIN predecessor node, return immediately. (?)
- Pruning: If one of its children has value lower than that of its best MAX predecessor node, return immediately. (?)
Properties of $\alpha - \beta$ Pruning

• Pruning does not affect final result

• The effectiveness of alpha-beta pruning is highly dependent on the order in which the successors are examined
  – Worthwhile to try to examine first the successors that are likely to be best
  – E.g., If the third successor “2” of node D has been generated first, the other two “14” and “5” can be pruned
Properties of $\alpha - \beta$ Pruning (cont.)

- If “perfect ordering” can be achieved
  - Time complexity: $O(b^{m/2})$
    - Effective branching factor becomes: $b^{1/2}$
    - Can double the depth of search within the time limit

- If “random ordering”
  - Time complexity $\approx O(b^{3m/4})$ for moderate $b$

- Still have to search all the way to terminal states for at least a portion of the search space
  - The depth is usually not practical
Properties of $\alpha - \beta$ Pruning (cont.)
Imperfect, Real-Time Decisions

• Not feasible to search all the way to terminal states in per move
  – When minimax search is adopted alone, or even when alpha-beta pruning is additionally involved
  – Moves must be made in a reasonable amount of time

• Shannon (1960) said
  – “…programs should cut off search earlier and apply a heuristic function to states in the search, effectively turning nonterminal nodes into terminal leaves…”
Imperfect, Real-Time Decisions (cont.)

• Minimax or alpha-beta altered in two ways

  – A heuristic evaluation function $Eval$ is used to replace the utility function
    • Give an estimate of the expected utility of the game from a given position
    • Judge the value of a position

  – A cutoff test is used to replace the terminal test
    • Decide when to apply $Eval$
    • Turn nonterminal nodes into terminal leaves
    • A fixed depth limit is used (often add quiescence search)
Evaluation Functions

• Criteria for good evaluation functions
  – Should order the terminal states in the same way as the true utility function
    • Avoid selecting suboptimal moves
  – Must not take too long to calculate
    • Time controls usually enforced
  – For nonterminal states, it should be strongly correlated with the actual chances of winning
    • Do not overestimate or underestimate too much
    • Chances here mean uncertainty, which is introduced by computational limits
      – A guess/prediction should be made
Evaluation Functions (cont.)

- **Method 1:** Most evaluation functions calculate and then combine various features of a state to give the estimation
  - E.g., the number of pawns possessed by each side in the chess game

  - Many states (with different board configurations) would have the same values of all features
    - States in the **same category** will win, draw, or lose proportionally/probabilistically

  \[
  (0.72 \times +1) + (0.20 \times -1) + (0.08 \times 0) = 0.52 \\
  \text{win} + \text{loss} + \text{draw} 
  \]

  - Too many categories required to calculate the expected values for evaluation functions, and hence too much experience required to estimate the probabilities
Evaluation Functions (cont.)

- **Method 2: Weighted linear function**
  - Directly compute separate numerical contributions from each feature and then combine them to find the total value for a state.

\[
\text{Eval}(s) = w_1 f_1(s) + w_2 f_2(s) + \cdots + w_j f_j(s) = \sum_{j=1}^{j} w_j f_j(s)
\]

- **Assumptions:**
  1. Features are independent on each other.
  2. Values of features won’t change with time.

- The material value for each piece in the chess game
  - E.g., a pawn has a value of 1, a bishop/knight for 3, a rook for 5, a queen for 9 etc.
Cutting Off Search

• When to call the heuristic evaluation function in order to appropriately cut off the search?

```latex
\textbf{if} \text{Cutoff-Test}(state, depth) \textbf{then return} \text{Eval}(state)
```

• Replace the “Terminal-Test” line in the algorithm
• The amount of search is controlled by setting a fixed depth limit such that the time constraint will not be violated
• Bookkeeping for the current node’s depth is needed

\textbf{Cutoff-Test}(state, depth)

• Return \textit{true} for all \textit{depth} greater than some fixed depth \textit{d}, and vice versa
• Return \textit{true} for all terminal states

• Iterative deepening search (IDS) can be applied here
  – Return the move selected by \textit{the deepest completed search}
Cutting Off Search: Problems

• Suppose when the program has searched to the depth limit and reached the following position

(a) Black an advantage of a knight and two pawns and will win the game
(b) Black will lose after white captures the queen

• A more sophisticated cutoff test (for quiescence) is needed!
Cutting Off Search: Quiescence

• A quiescent position is one which is unlikely to exhibit wild swings in value in the near future

• Nonquiescent positions can be expanded further until quiescent positions are reached
  – Called quiescence search
    • Search for certain types of moves
    • E.g., search for “capture moves”
Deterministic Games in Practices

• Checkers
  – 1994, the computer defeated the human world champion

• Chess
  – 1997, Deep blue defeated the human world champion
    • Can seek 200 million positions per sec (almost 40 plies)

• Othello
  – Computers are superior

• Go
  – Humans are superior
Nondeterministic Games: Backgammon

• Games that combine luck and skill
  – Dice are rolled at the beginning of a player’s turn to determine the legal moves
  – E.g., Backgammon

1. **Goal of the game:** move all one’s pieces off the board
2. White moves clockwise toward 25
   Black moves counterclockwise toward 0
3. A piece can move to any position unless there are multiple opponent pieces there
4. If the position to be move to has only one opponent, the opponent will be captured and restarted over
5. When one’s all pieces are in his home board, the pieces can be moved off the board

…

When white has rolled 6-5, it must choose among four legal moves:
(5-10,5-11),(5-11,19-24),(5-10,10-16) and (5-11,11-16)
Nondeterministic Games: Backgammon (cont.)

• A game tree includes chance nodes

If two dice used:
- 21 distinct rolls
  - 15 \( \binom{6}{2} \) with probabilities 1/18
  - 6 \( \binom{6}{1} \) with probabilities 1/36
Nondeterministic Games in General

- Chance introduced by dice, card-shuffling
  - E.g., a simplified example with coin-flipping
Algorithm for Nondeterministic Games

• Expectiminimax gives perfect play
  – Just like minimax, except chance nodes must be also handled

\[
\text{expectiminimax}(n) = \\
\begin{aligned}
\text{Utility}(n) & \quad \text{if } n \text{ is a terminal state} \\
\max_{s \in \text{Successor}(n)} \text{expectiminimax}(s) & \quad \text{if } n \text{ is a MAX node} \\
\min_{s \in \text{Successor}(n)} \text{expectiminimax}(s) & \quad \text{if } n \text{ is a MIN node} \\
\sum_{s \in \text{Successor}(n)} P(s) \cdot \text{expectiminimax}(s) & \quad \text{if } n \text{ is a chance node}
\end{aligned}
\]
Pruning in Nondeterministic Game Trees

• A version of \( \alpha - \beta \) pruning is possible
Pruning in Nondeterministic Game Trees (cont.)

- A version of $\alpha - \beta$ pruning is possible
Pruning in Nondeterministic Game Trees (cont.)

• A version of $\alpha - \beta$ pruning is possible
Pruning in Nondeterministic Game Trees (cont.)

- A version of $\alpha - \beta$ pruning is possible

![Game Tree Diagram](image)
Pruning in Nondeterministic Game Trees (cont.)

- A version of $\alpha - \beta$ pruning is possible
Pruning in Nondeterministic Game Trees (cont.)

• A version of $\alpha - \beta$ pruning is possible
Pruning in Nondeterministic Game Trees (cont.)

• A version of $\alpha - \beta$ pruning is possible
Pruning in Nondeterministic Game Trees (cont.)

- A version of $\alpha - \beta$ pruning is possible
Pruning with Bounds

• More pruning if we can bound the leaf values
Pruning with Bounds (cont.)

• More pruning if we can bound the leaf values
Pruning with Bounds (cont.)

- More pruning if we can bound the leaf values
Pruning with Bounds (cont.)

- More pruning if we can bound the leaf values
Pruning with Bounds (cont.)

- More pruning if we can bound the leaf values
Pruning with Bounds (cont.)

• More pruning if we can bound the leaf values

- Save 2/7 operations than the previously unconstrained approach
Nondeterministic Games in Practice

- For backgammon with two dice rolled
  - 20 legal moves on average (could be more than 4,000 for 1-1 roll)
    - Branching factor $b \approx 20$
  - 21 possible rolls
    - Number of distinct rolls $n=21$
  - E.g., if depth=4
    - $20 \times (21 \times 20)^3 \approx 1.2 \times 10^9$

- $\alpha - \beta$ pruning is much less effective here
Digression: Exact Value Do Matter

• Behavior is preserved only by positive linear transformation of evaluation function \( \text{Eval} \)
  – Hence, \( \text{Eval} \) should be proportional to the expected payoff
Games of Imperfect Information

• E.g., card game, where opponent’s initial cards are unknown
  – Typically we can calculate a probability for each possible deal
  – Seems just like having one big dice roll at the beginning of the game

• Idea: compute the minimax value of each action in each deal, then choose the action with highest expected value over all deals
  – Special case: if an action is optimal for all deals, it’s optimal

• GIB, current best bridge program, approximate this idea by
  – Generating 100 deals consistent with bidding information
  – Picking the action that wins most tricks on average
Example

• Four-card bridge/whist/hearts hand, Max to play first
Example (cont.)

- Four-card bridge/whist/hearts hand, $\text{Max}$ to play first
Example (cont.)

- Four-card bridge/whist/hearts hand, $\text{Max}$ to play first
Example (cont.)

```
+1.5
MAX

+1
0.5
MAX 0.5

+2
MIN 0.5

+1
0.5 MIN 0.5

+1.5
MAX

+1
0.5
MAX 0.5

+2
MIN 0.5

+1
0.5 MIN 0.5

+1
0.5 MIN 0.5

+2
MIN 0.5
```

AI – Berlin Chen 68
Example (cont.)

- It is a good idea to play a card that will help one discover things about one’s opponent’s cards or that will tell one’s partner about one’s own cards.

- It is best to give away as little information to the opponent as possible, and often the best way to do this is to act unpredictable.