Linear Algebra
Quiz 4 (Brief Solution)
11:10 a.m. - 12:10 a.m., December 18, 2015

Note: You have to answer the questions with supporting explanations (i.e., show all your work) if needed.

1. Show that the three vectors \( v_1 = (0, 3, 1, -1) \), \( v_2 = (6, 0, 5, 1) \) and \( v_3 = (4, -7, 1, 3) \) form a linearly dependent set in \( \mathbb{R}^4 \). (15%)

Ans.: Let \( k_1 v_1 + k_2 v_2 + k_3 v_3 = 0 \)

It is easy to find (with elementary row operations) a set of not all zero values of \( k_1, k_2, k_3 \) (i.e., a nontrivial solution), for example \( k_1 = 7, k_2 = -2, k_3 = 3 \), such that \( k_1 v_1 + k_2 v_2 + k_3 v_3 = 0 \).

2. Find the coordinate vector of \( \mathbf{p} = 2 - x - x^2 \) relative to the following three basis vectors: \( \mathbf{p}_1 = 1 + x \), \( \mathbf{p}_2 = 1 + x^2 \) and \( \mathbf{p}_3 = x + x^2 \). (15%)

Ans.: Let \( c_1 \mathbf{p}_1 + c_2 \mathbf{p}_2 + c_3 \mathbf{p}_3 = \mathbf{p} \). It is easy to find that \( c_1 = 1, c_2 = 1, c_3 = -2 \).

Therefore, the coordinate vector of \( \mathbf{p} \) relative to \( \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \) is \([1, 1, -2]^T\).

3. Given a matrix \( A \) shown below:

\[
A = \begin{bmatrix}
1 & 1 & 2 & 4 & 2 \\
2 & 1 & 2 & 3 & 1 \\
1 & 2 & 4 & 9 & 2 \\
6 & 3 & 6 & 9 & 3
\end{bmatrix}
\]

(i) Find a basis for the column space of \( A \) consisting entirely of column vectors of \( A \). (10%)

(ii) Find a basis for the row space of \( A \) consisting entirely of row vectors of \( A \). (10%)

(iii) Find a basis for the null space of \( A \). (10%)

(iv) What is the rank of \( A \)? and what is the dimension of the null space of \( A \)? (10%)

Ans.: 

(i) basis for the column space = \{\( [1, 2, 1, 6]^T, [1, 1, 2, 3]^T, [2, 1, 2, 3]^T \)\}

(ii) basis for the null space = \{\( [1, 1, 2, 4, 2], [2, 1, 2, 3, 1], [1, 2, 4, 9, 2] \)\}

(iii) basis for the null space = \{\( [1, -5, 0, 1, 0], [0, -2, 1, 0, 0] \)\}

(iv) the rank of \( A \) = 3; the dimension of the null space of \( A \) = 2

4. Consider the bases \( B = \{\mathbf{u}_1, \mathbf{u}_2\} \) and \( B' = \{\mathbf{v}_1, \mathbf{v}_2\} \) for \( \mathbb{R}^2 \), where

\[
\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.
\]

(i) Find the transition (change-of-coordinates) matrix from \( B \) to \( B' \). (10%)

(ii) Find the transition (change-of-coordinates) matrix from \( B' \) to \( B \). (10%)

(iii) Given that \( [\mathbf{w}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \), compute \( [\mathbf{w}]_{B'} \). (10%)
Ans.:

(i) $P_{B \rightarrow B} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$

(ii) $P_{B' \rightarrow B} = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$

(iii) $[w]_{B'} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$